The z-transform

See Oppenheim and Schafer, Second Edition pages 94–139, or First Edition pages 149–201.

1 Introduction

The z-transform of a sequence $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

The z-transform can also be thought of as an operator $\mathcal{Z}\{\cdot\}$ that transforms a sequence to a function:

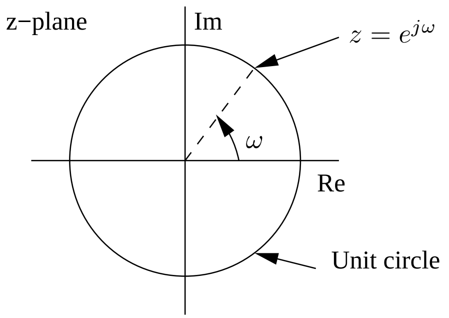
$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z).$$

In both cases z is a continuous complex variable.

We may obtain the Fourier transform from the z-transform by making the substitution $z = e^{j\omega}$. This corresponds to restricting $|z| = 1$. Also, with $z = re^{j\omega}$,

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n}.$$

That is, the z-transform is the Fourier transform of the sequence $x[n]r^{-n}$. For $r = 1$ this becomes the Fourier transform of $x[n]$. The Fourier transform therefore corresponds to the z-transform evaluated on the unit circle:



The inherent periodicity in frequency of the Fourier transform is captured naturally under this interpretation.

The Fourier transform does not converge for all sequences — the infinite sum may not always be finite. Similarly, the z-transform does not converge for all sequences or for all values of z. The set of values of z for which the z-transform converges is called the **region of convergence (ROC)**.

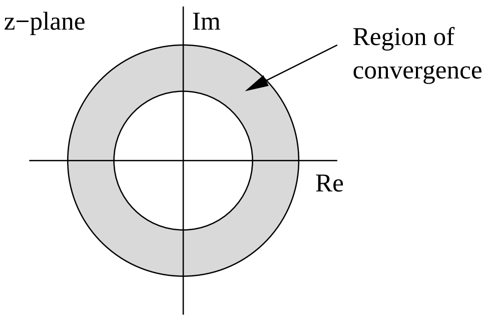
The Fourier transform of $x[n]$ exists if the sum $\sum_{n=-\infty}^{\infty} |x[n]|$ converges. However, the z-transform of $x[n]$ is just the Fourier transform of the sequence $x[n]r^{-n}$. The z-transform therefore exists (or converges) if

$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty.$$

This leads to the condition

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

for the existence of the z-transform. The ROC therefore consists of a ring in the z-plane:



In specific cases the inner radius of this ring may include the origin, and the outer radius may extend to infinity. If the ROC includes the unit circle $|z| = 1$, then the Fourier transform will converge.

Most useful z-transforms can be expressed in the form

$$X(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials in z. The values of z for which $P(z) = 0$ are called the **zeros** of $X(z)$, and the values with $Q(z) = 0$ are called the **poles**. The zeros and poles completely specify $X(z)$ to within a multiplicative constant.

Example: right-sided exponential sequence
Consider the signal $x[n] = a^n u[n]$. This has the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

Convergence requires that

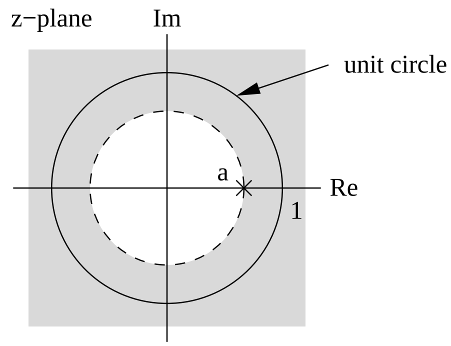
$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty,$$

which is only the case if $|az^{-1}| < 1$, or equivalently $|z| > |a|$. In the ROC, the

series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|,$$

since it is just a geometric series. The z-transform has a region of convergence for any finite value of a.



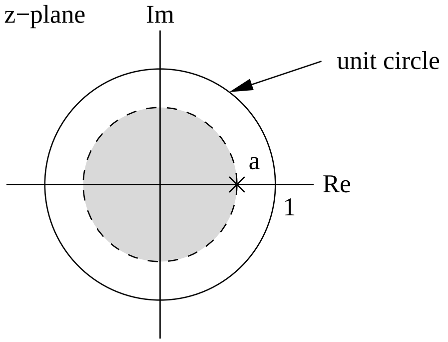
The Fourier transform of $x[n]$ only exists if the ROC includes the unit circle, which requires that $|a| < 1$. On the other hand, if $|a| > 1$ then the ROC does not include the unit circle, and the Fourier transform does not exist. This is consistent with the fact that for these values of a the sequence $a^n u[n]$ is exponentially growing, and the sum therefore does not converge.

Example: left-sided exponential sequence
Now consider the sequence $x[n] = -a^n u[-n-1]$. This sequence is left-sided because it is nonzero only for $n \leq -1$. The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$
$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n.$$

For $|a^{-1}z| < 1$, or $|z| < |a|$, the series converges to

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| < |a|.$$



Note that the expression for the z-transform (and the pole zero plot) is exactly the same as for the right-handed exponential sequence — *only the region of convergence is different*. Specifying the ROC is therefore critical when dealing with the z-transform.

Example: sum of two exponentials
The signal $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$ is the sum of two real exponentials. The z-transform is

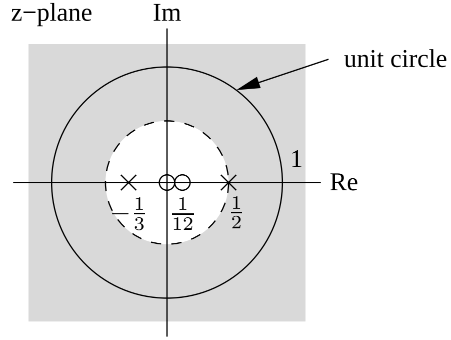
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n. \end{aligned}$$

From the example for the right-handed exponential sequence, the first term in this sum converges for $|z| > 1/2$, and the second for $|z| > 1/3$. The combined transform $X(z)$ therefore converges in the intersection of these regions,

namely when $|z| > 1/2$. In this case

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}.$$

The pole-zero plot and region of convergence of the signal is



Example: finite length sequence
The signal

$$x[n] = \begin{cases} a^n & 0 \le n \le N - 1 \\ 0 & \text{otherwise} \end{cases}$$

has z-transform

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}. \end{aligned}$$

Since there are only a finite number of nonzero terms the sum always converges when az^{-1} is finite. There are no restrictions on a ($|a| < \infty$), and the ROC is the entire z-plane with the exception of the origin $z = 0$ (where the terms in the sum are infinite). The N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \ldots, N - 1,$$

since these values satisfy the equation $z^N = a^N$. The zero at $k = 0$ cancels the pole at $z = a$, so there are no poles except at the origin, and the zeros are at

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 1, \ldots, N-1.$$

2 Properties of the region of convergence

The properties of the ROC depend on the nature of the signal. Assuming that the signal has a finite amplitude and that the z-transform is a rational function:

- The ROC is a ring or disk in the z-plane, centered on the origin ($0 \leq r_R < |z| < r_L \leq \infty$).
- The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform includes the unit circle.
- The ROC cannot contain any poles.
- If $x[n]$ is finite duration (ie. zero except on finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$), then the ROC is the entire z-plane except perhaps at $z = 0$ or $z = \infty$.
- If $x[n]$ is a right-sided sequence then the ROC extends outward from the outermost finite pole to infinity.
- If $x[n]$ is left-sided then the ROC extends inward from the innermost nonzero pole to $z = 0$.
- A two-sided sequence (neither left nor right-sided) has a ROC consisting of a ring in the z-plane, bounded on the interior and exterior by a pole (and not containing any poles).
- The ROC is a connected region.

3 The inverse z-transform

Formally, the inverse z-transform can be performed by evaluating a Cauchy integral. However, for discrete LTI systems simpler methods are often sufficient.

3.1 Inspection method

If one is familiar with (or has a table of) common z-transform pairs, the inverse can be found by inspection. For example, one can invert the z-transform

$$X(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right), \qquad |z| > \frac{1}{2},$$

using the z-transform pair

$$a^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \qquad \text{for } |z| > |a|.$$

By inspection we recognise that

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

Also, if $X(z)$ is a sum of terms then one may be able to do a term-by-term inversion by inspection, yielding $x[n]$ as a sum of terms.

3.2 Partial fraction expansion

For any rational function we can obtain a partial fraction expansion, and identify the z-transform of each term. Assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

It is always possible to factor $X(z)$ as

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})},$$

where the c_k's and d_k's are the nonzero zeros and poles of $X(z)$.

- If $M < N$ and the poles are all first order, then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

  In this case the coefficients A_k are given by

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z=d_k}.$$

- If $M \geq N$ and the poles are all first order, then an expansion of the form

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

  can be used, and the B_r's be obtained by long division of the numerator by the denominator. The A_k's can be obtained using the same equation as for $M < N$.

- The most general form for the partial fraction expansion, which can also deal with multiple-order poles, is

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}.$$

  Ways of finding the C_m's can be found in most standard DSP texts.

The terms $B_r z^{-r}$ correspond to shifted and scaled impulse sequences, and invert to terms of the form $B_r \delta[n - r]$. The fractional terms

$$\frac{A_k}{1 - d_k z^{-1}}$$

correspond to exponential sequences. For these terms the ROC properties must be used to decide whether the sequences are left-sided or right-sided.

Example: inverse by partial fractions

Consider the sequence $x[n]$ with z-transform

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}, \qquad |z| > 1.$$

Since $M = N = 2$ this can be expressed as

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

The value B_0 can be found by long division:

$$\begin{array}{r@{\,}l} & \;2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & \overline{)\; z^{-2} + 2z^{-1} + 1} \\ & \;\; \underline{z^{-2} - 3z^{-1} + 2} \\ & \qquad\quad 5z^{-1} - 1 \end{array}$$

so

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}.$$

The coefficients A_1 and A_2 can be found using

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z=d_k},$$

so

$$A_1 = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1}}\bigg|_{z^{-1}=2} = \frac{1 + 4 + 4}{1 - 2} = -9$$

and

$$A_2 = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1}}\bigg|_{z^{-1}=1} = \frac{1 + 2 + 1}{1/2} = 8.$$

Therefore

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

Using the fact that the ROC is $|z| > 1$, the terms can be inverted one at a time by inspection to give

$$x[n] = 2\delta[n] - 9(1/2)^n u[n] + 8u[n].$$

3.3 Power series expansion

If the z-transform is given as a power series in the form

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \ldots + x[-2]z^2 + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots, \end{aligned}$$

then any value in the sequence can be found by identifying the coefficient of the appropriate power of z^{-1}.

Example: finite-length sequence
The z-transform

$$X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})$$

can be multiplied out to give

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}.$$

By inspection, the corresponding sequence is therefore

$$x[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently

$$x[n] = 1\delta[n+2] - \frac{1}{2}\delta[n+1] - 1\delta[n] + \frac{1}{2}\delta[n-1].$$

Example: power series expansion
Consider the z-transform

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|.$$

Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, gives

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}.$$

The corresponding sequence is therefore

$$x[n] = \begin{cases} (-1)^{n+1} \frac{a^n}{n} & n \geq 1 \\ 0 & n \leq 0. \end{cases}$$

Example: power series expansion by long division
Consider the transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|.$$

Since the ROC is the exterior of a circle, the sequence is right-sided. We therefore divide to get a power series in powers of z^{-1}:

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & 1 - az^{-1} \\ \hline & az^{-1} \\ & az^{-1} - a^2z^{-2} \\ \hline & a^2z^{-2} + \cdots \end{array}$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots.$$

Therefore $x[n] = a^n u[n]$.

Example: power series expansion for left-sided sequence
Consider instead the z-transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| < |a|.$$

Because of the ROC, the sequence is now a left-sided one. Thus we divide to obtain a series in powers of z:

$$\begin{array}{r} -a^{-1}z - a^{-2}z^2 - \cdots \\ -a + z \overline{\big) \, z \qquad\qquad\qquad} \\ \underline{z - a^{-1}z^2} \\ az^{-1} \end{array}$$

Thus $x[n] = -a^n u[-n-1]$.

4 Properties of the z-transform

In this section, if $X(z)$ denotes the z-transform of a sequence $x[n]$ and the ROC of $X(z)$ is indicated by R_x, then this relationship is indicated as

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad \text{ROC} = R_x.$$

Furthermore, with regard to nomenclature, we have two sequences such that

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z), \qquad \text{ROC} = R_{x_1}$$
$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z), \qquad \text{ROC} = R_{x_2}.$$

4.1 Linearity

The linearity property is as follows:

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC contains} R_{x_1} \cap R_{x_1}.$$

4.2 Time shifting

The time-shifting property is as follows:

$$x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z), \qquad \text{ROC} = R_x.$$

(The ROC may change by the possible addition or deletion of $z = 0$ or $z = \infty$.) This is easily shown:

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n} = \sum_{m=-\infty}^{\infty} x[m] z^{-(m+n_0)}$$
$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m} = z^{-n_0} X(z).$$

Example: shifted exponential sequence
Consider the z-transform

$$X(z) = \frac{1}{z - \frac{1}{4}}, \qquad |z| > \frac{1}{4}.$$

From the ROC, this is a right-sided sequence. Rewriting,

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}} = z^{-1} \left(\frac{1}{1 - \frac{1}{4}z^{-1}} \right), \qquad |z| > \frac{1}{4}.$$

The term in brackets corresponds to an exponential sequence $(1/4)^n u[n]$. The factor z^{-1} shifts this sequence one sample to the right. The inverse z-transform is therefore

$$x[n] = (1/4)^{n-1} u[n-1].$$

Note that this result could also have been easily obtained using a partial fraction expansion.

4.3 Multiplication by an exponential sequence

The exponential multiplication property is

$$z_0^n x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z/z_0), \qquad \text{ROC} = |z_0|R_x,$$

where the notation $|z_0|R_x$ indicates that the ROC is scaled by $|z_0|$ (that is, inner and outer radii of the ROC scale by $|z_0|$). All pole-zero locations are similarly scaled by a factor z_0: if $X(z)$ had a pole at $z = z_1$, then $X(z/z_0)$ will have a pole at $z = z_0 z_1$.

- If z_0 is positive and real, this operation can be interpreted as a shrinking or expanding of the z-plane — poles and zeros change along radial lines in the z-plane.

- If z_0 is complex with unit magnitude ($z_0 = e^{j\omega_0}$) then the scaling operation corresponds to a rotation in the z-plane by and angle ω_0. That is, the poles and zeros rotate along circles centered on the origin. This can be interpreted as a shift in the frequency domain, associated with modulation in the time domain by $e^{j\omega_0 n}$. If the Fourier transform exists, this becomes

$$e^{j\omega_0 n} x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).$$

Example: exponential multiplication
The z-transform pair

$$u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

can be used to determine the z-transform of $x[n] = r^n \cos(\omega_0 n) u[n]$. Since $\cos(\omega_0 n) = 1/2 e^{j\omega_0 n} + 1/2 e^{-j\omega_0 n}$, the signal can be rewritten as

$$x[n] = \frac{1}{2}(re^{j\omega_0})^n u[n] + \frac{1}{2}(re^{-j\omega_0})^n u[n].$$

From the exponential multiplication property,

$$\frac{1}{2}(re^{j\omega_0})^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1/2}{1-re^{j\omega_0}z^{-1}}, \qquad |z| > r$$
$$\frac{1}{2}(re^{-j\omega_0})^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1/2}{1-re^{-j\omega_0}z^{-1}}, \qquad |z| > r,$$

so

$$\begin{aligned} X(z) &= \frac{1/2}{1-re^{j\omega_0}z^{-1}} + \frac{1/2}{1-re^{-j\omega_0}z^{-1}}, \qquad |z| > r \\ &= \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}, \qquad |z| > r. \end{aligned}$$

4.4 Differentiation

The differentiation property states that

$$nx[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x.$$

This can be seen as follows: since

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

we have

$$-z\frac{dX(z)}{dz} = -z\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\}.$$

Example: second order pole
The z-transform of the sequence

$$x[n] = na^n u[n]$$

can be found using

$$a^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad |z| > a,$$

to be

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > a.$$

4.5 Conjugation

This property is

$$x^*[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \qquad \text{ROC} = R_x.$$

4.6 Time reversal

Here

$$x^*[-n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X^*(1/z^*), \qquad \text{ROC} = \frac{1}{R_x}.$$

The notation $1/R_x$ means that the ROC is inverted, so if R_x is the set of values such that $r_R < |z| < r_L$, then the ROC is the set of values of z such that $1/r_l < |z| < 1/r_R$.

Example: time-reversed exponential sequence
The signal $x[n] = a^{-n}u[-n]$ is a time-reversed version of $a^n u[n]$. The z-transform is therefore

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

4.7 Convolution

This property states that

$$x_1[n] * x_2[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC contains} R_{x_1} \cap R_{x_2}.$$

Example: evaluating a convolution using the z-transform
The z-transforms of the signals $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$ are

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}, \qquad |z| > |a|$$

and

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}, \qquad |z| > 1.$$

For $|a| < 1$, the z-transform of the convolution $y[n] = x_1[n] * x_2[n]$ is

$$Y(z) = \frac{1}{(1-az^{-1})(1-z^{-1})} = \frac{z^2}{(z-a)(z-1)}, \qquad |z| > 1.$$

Using a partial fraction expansion,

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \qquad |z| > 1,$$

so

$$y[n] = \frac{1}{1-a}(u[n] - a^{n+1}u[n]).$$

4.8 Initial value theorem

If $x[n]$ is zero for $n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z).$$

Some common z-transform pairs are:

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$

4.9 Relationship with the Laplace transform

Continuous-time systems and signals are usually described by the Laplace transform. Letting $z = e^{sT}$, where s is the complex Laplace variable

$$s = d + j\omega,$$

we have

$$z = e^{(d+j\omega)T} = e^{dT} e^{j\omega T}.$$

Therefore

$$|z| = e^{dT} \quad \text{and} \quad \sphericalangle z = \omega T = 2\pi f/f_s = 2\pi\omega/\omega_s,$$

where ω_s is the sampling frequency. As ω varies from ∞ to ∞, the s-plane is mapped to the z-plane:

- The $j\omega$ axis in the s-plane is mapped to the unit circle in the z-plane.
- The left-hand s-plane is mapped to the inside of the unit circle.
- The right-hand s-plane maps to the outside of the unit circle.