EEE4001F: Digital Signal Processing

Class Test 1

20 March 2008

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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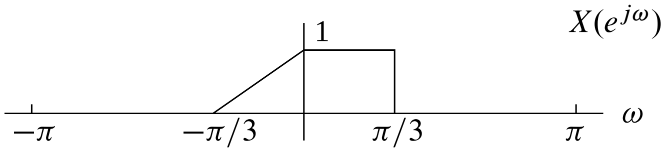
1. (5 marks) A sequence $x[n]$ has a zero-phase DTFT $X(e^{j\omega})$ given below:



Sketch the DTFT of the sequence $y[n] = x[-n]e^{-j\pi n/3}$.

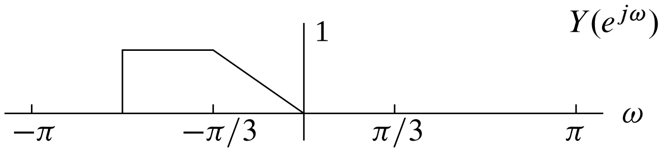
If $x[n] \Longleftrightarrow X(e^{j\omega})$, then by the time reversal property we know that $x[-n] \Longleftrightarrow X(e^{-j\omega})$. Applying frequency shift gives

$$x[-n]e^{j\omega_0 n} \Longleftrightarrow X(e^{-j(\omega-\omega_0)}),$$

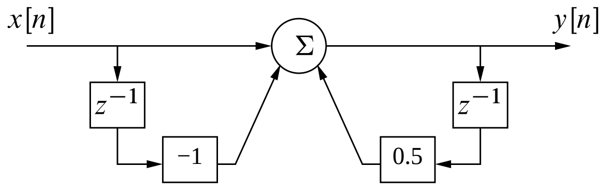
which with $\omega_0 = -\pi/3$ is

$$x[-n]e^{-j\pi n/3} \Longleftrightarrow X(e^{-j(\omega+\pi/3)}).$$

Thus $Y(e^{j\omega}) = X(e^{-j(\omega+\pi/3)})$:



2. (5 marks) Consider the following LTI system:



Determine a closed-form expression for the response $y[n]$ of this system to the following input signal:

$$x[n] = \begin{cases} 1 & n \geq 4 \\ 0 & \text{otherwise} \end{cases}$$

if the system is causal and initially at rest.

The difference equation for the system is

$$y[n] = 0.5y[n-1] + x[n] - x[n-1],$$

so

$$Y(z) = 0.5z^{-1}Y(z) + X(z) - z^{-1}X(z)$$

and

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 0.5z^{-1}} = \frac{z-1}{z-0.5}.$$

For causality the ROC must be $|z| > 0.5$.

The input $x[n] = u[n-4]$ has z-transform

$$X(z) = \frac{z^{-1}}{1 - z^{-4}}$$

with ROC $|z| > 1$, so the output is

$$Y(z) = \frac{1 - z^{-1}}{1 - 0.5z^{-1}} \frac{z^{-4}}{1 - z^{-1}} = \frac{z^{-4}}{1 - 0.5z^{-1}}$$

The ROC of $Y(z)$ is either $|z| < 0.5$ or $|z| > 0.5$. However, this ROC must contain the intersection of the ROCs of $X(z)$ and $H(z)$, which is $|z| > 1$. Thus the ROC for $Y(z)$ is $|z| > 0.5$ and the inverse is $y[n] = (0.5)^{n-4}u[n-4]$.

3. (5 marks) Consider the following discrete-time signal $x[n]$:

$$x[n] = \begin{cases} n+1 & 0 \le n \le 3 \\ 4 & n \ge 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the z-transform $X(z)$ of $x[n]$, and represent it as a ratio of polynomials in z^{-1}.

(b) What is the region of convergence (ROC) of this z-transform?

(a) The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = 1z^{-0} + 2z^{-1} + 3z^{-2} + 4z^{-3} + \sum_{n=4}^{\infty} 4z^{-n}$$
$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 4\sum_{n=4}^{\infty} z^{-n}$$

Since

$$\sum_{n=4}^{\infty} z^{-n} = z^{-4} + z^{-5} + \cdots = z^{-4}(1 + z^{-1} + \cdots) = z^{-4}\sum_{n=0}^{\infty} z^{-n},$$

and

$$\sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1-z^{-1}}$$

for $|z^{-1}| < 1$, we have

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 4\frac{z^{-4}}{1-z^{-1}}.$$

This can be written in the form

$$X(z) = \frac{1 + z^{-1} + z^{-2} + z^{-3}}{1 - z^{-1}}.$$

(b) The previous expression is valid as long as $|z^{-1}| < 1$. Letting $z = re^{j\omega}$ we see that $|z^{-1}| = |r^{-1}e^{-j\omega}| = r^{-1} = |z|^{-1}$, so the region of convergence is $|z|^{-1} < 1$ or $|z| > 1$.

4. (5 marks) Consider the continuous-time signal

$$x(t) = \sin(400\pi t + \pi).$$

The discrete-time signal $x[n]$ is obtained by sampling $x(t)$ at $t = n/f_s$ with a sampling frequency $f_s = 1000$ Hz. Which one of the following continuous-time signals will yield the same sample values when sampled at the same sampling instants? Show and motivate your calculations.

(a) $\sin(600\pi t)$

(b) $-\sin(1000\pi t)$

(c) $\sin(1400\pi t)$

(d) $\sin(1600\pi t)$

The discrete time signal obtained by sampling $x(t)$ is

$$x[n] = x(n/f_s) = \sin\left(\frac{400\pi}{1000}n + \pi\right)$$

which has frequency content at $\omega = \pm 400\pi/1000 + 2\pi k$, or $\omega = \pm 3\pi/5 + 2\pi k$ radians per sample.

Sampling the signal in (a) in the same way gives $x_a[n] = \sin((600\pi/1000)n)$, which contains the frequencies $\omega = \pm 600\pi/1000 + 2\pi k$, or $\omega = \pm 3\pi/5 + 2\pi k$ radians per sample. Since $x[n]$ contains no component at this frequency, the discrete signals cannot be the same. Similarly, the signal in (b) only contains the frequency $\omega = 0 + 2\pi k$ radians per sample, and the one in (c) only contains $\omega = \pm 3\pi/5 + 2\pi k$ radians per sample. They also cannot be the same.

Since (d) is the only option, it must be the solution. This can be proven:

$$x[n] = \frac{1}{2j}\left(e^{j[(2\pi/5)n+\pi]} - e^{-j[(2\pi/5)n-\pi]}\right) = \frac{1}{2j}\left(-e^{j(2\pi/5)n} + e^{-j(2\pi/5)n}\right)$$

However,

$$x_d[n] = \sin\left(\frac{1600\pi}{1000}n\right) = \frac{1}{2j}\left(e^{j(8\pi/5)n} - e^{-j(8\pi/5)n}\right)$$
$$= \frac{1}{2j}\left(e^{-j(2\pi/5)n} - e^{j(2\pi/5)n}\right) = x[n].$$

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \|\omega\| < \omega_c \\ 0 & \omega_c < \|\omega\| \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$