# EEE4001F: Digital Signal Processing

## Class Test 2

22 April 2010

# SOLUTIONS

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**Name:**

**Student number:**

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### Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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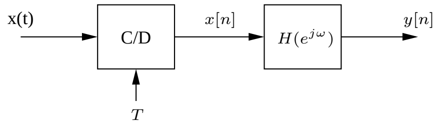
1. (5 marks) Consider the system below




where $T = 0.001$s and

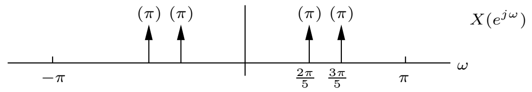
$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq 0.5\pi \\ 0 & \text{otherwise} \end{cases}$$

for $-\pi \leq \omega \leq \pi$. Find the output $y[n]$ if the input is $x(t) = \cos(400\pi t) + \cos(600\pi t)$.

The discretised input is

$$x[n] = x(nT) = \cos(\frac{400}{1000}\pi n) + \cos(\frac{600}{1000}\pi n) = \cos\left(\frac{2\pi}{5}n\right) + \cos\left(\frac{3\pi}{5}n\right)$$
$$= \frac{1}{2}e^{j\frac{2\pi}{5}n} + \frac{1}{2}e^{-j\frac{2\pi}{5}n}\frac{1}{2}e^{j\frac{3\pi}{5}n} + \frac{1}{2}e^{-j\frac{3\pi}{5}n}.$$

In the frequency domain this is




and the filter removes the two impulses at frequencies $\omega = \pm\frac{3\pi}{5}$, and hence the $\cos\left(\frac{3\pi}{5}n\right)$ term. The output is therefore just the remaining term

$$y[n] = \cos\left(\frac{2\pi}{5}n\right).$$

2. (5 marks) Consider the following discrete-time signals $x[n]$ and $y[n]$:

$$x[n] = 0.2\cos(0.2\pi n) \quad \text{and} \quad y[n] = 0.2\sin(0.2\pi n).$$

(a) Show that the 10-point DFT of $x[n]$ is $X[k] = \delta[k-1] + \delta[k-9]$ over the range $k = 0, \dots, 9$.

(b) Assuming that the 10-point DFT of $y[n]$ is $Y[k] = -j(\delta[k-1] - \delta[k-9])$, use the DFT to determine a *closed-form* expression for the 10-point circular convolution of $x[n]$ and $y[n]$.

(a) The DFT is as follows:

$$X[k] = \sum_{n=0}^{9} 0.2\cos\left(\frac{2\pi}{10}n\right) e^{-j\frac{2\pi}{10}kn} = \frac{0.2}{2}\sum_{n=0}^{9}(e^{j\frac{2\pi}{10}n} + e^{-j\frac{2\pi}{10}n})e^{-j\frac{2\pi}{10}kn}$$
$$= \frac{1}{10}\sum_{n=0}^{9}\left(e^{-j\frac{2\pi}{10}(k-1)}\right)^n + \frac{1}{10}\sum_{n=0}^{9}\left(e^{-j\frac{2\pi}{10}(k+1)}\right)^n$$

Over the range $k = 0, \dots, 9$ the first term equals zero except when $k = 1$, in which case it equals 1. The second term is similar: one if $k = 9$ and zero otherwise. Thus

$$X[k] = \delta[k-1] + \delta[k-9].$$

(b) Ten-point circular convolution in time corresponds to multiplication in the 10-point DFT domain. Thus we need to take the 10-point DFTs of the signals, multiply them, and find the 10-point inverse DFT of the result.

If $w[n]$ is the 10-point circular convolution of $x[n]$ with $y[n]$, then for $k = 0, \dots, 9$ we have

$$W[k] = X[k]Y[k] = -j(\delta[k-1] - \delta[k-9])(\delta[k-1] + \delta[k-9])$$
$$= -j(\delta[k-1] - \delta[k-9]).$$

We observe that in this case $W[k] = Y[k]$, so the convolution result must be $w[n] = 0.2\sin(0.2\pi n)$.

3. (5 marks) A stable system is characterised by the following LCCDE:

$$y[n+2] - y[n+1] + \frac{1}{2}y[n] = x[n+1].$$

(a) Draw a pole-zero plot of the system.

(b) Roughly sketch the magnitude response of the system.

(c) Assuming the system response represents a band-pass filter at a frequency of $\pi/4$ radians/sample, what is the centre frequency of the passband if an analog signal is sampled at 12kHz before filtering?

(a) Taking the z-transform of the LCCDE gives

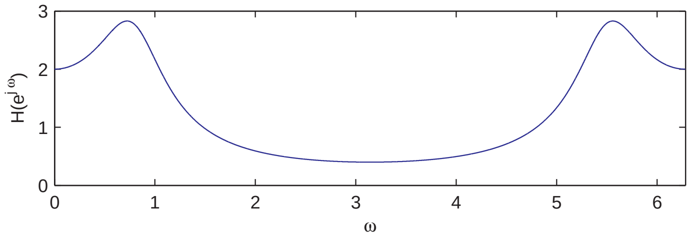
$$z^2Y(z) - zY(z) + \frac{1}{2}Y(z) = zX(z),$$

so

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z^2 - z + \frac{1}{2}}.$$

There is one zero in the numerator at $z = 0$. The roots of the denominator occur at $z = \frac{1}{2} \pm \frac{1}{2}j$. The sketch follows.

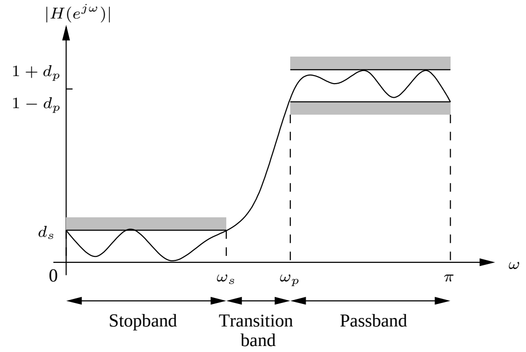
(b) Using graphical methods the magnitude response can be shown to be




(c) For a 12kHz sampling rate the Nyquist frequency is 6kHz and corresponds to $\omega = \pi$. The discrete frequency $\omega = \pi/4$ therefore relates to $6/4 = 1.5$kHz, and this is the center frequency of the analog system.

4. (5 marks) A particular DSP system is sampled at 48kHz, and requires a highpass filter with a passband ripple of 0.1dB, a transition band of 200Hz, stopband attenuation of 60dB, and a cutoff frequency of 1200Hz. Sketch the appropriate design constraints that the filter must satisfy, specifying parameter values where appropriate. Your frequency axis should be in units of radians per sample.

A prototype highpass filter is as follows:




For a sampling rate of 48kHz the highest frequency that can be represented is 24kHz, and this corresponds to $\omega = \pi$ rad/sample.

The filter must attenuate frequencies of less than 1200Hz by at least 60dB. Thus we must have $\omega_s = \frac{\pi}{24000}1200 = \pi/20$ rad/sample and $d_s = -60$dB. Similarly, $\omega_p = \frac{\pi}{24000}(1200+200)$ rad/sample and $d_p = 0.1$dB.

# Fourier transform properties

| Sequences $x[n]$, $y[n]$ | Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$ | Property |
|---|---|---|
| $ax[n] + by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ | Linearity |
| $x[n-n_d]$ | $e^{-j\omega n_d}X(e^{j\omega})$ | Time shift |
| $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ | Frequency shift |
| $x[-n]$ | $X(e^{-j\omega})$ | Time reversal |
| $nx[n]$ | $j\frac{dX(e^{j\omega})}{d\omega}$ | Frequency diff. |
| $x[n] * y[n]$ | $X(e^{-j\omega})Y(e^{-j\omega})$ | Convolution |
| $x[n]y[n]$ | $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ | Modulation |

# Common Fourier transform pairs

| Sequence | Fourier transform |
|---|---|
| $\delta[n]$ | $1$ |
| $\delta[n-n_0]$ | $e^{-j\omega n_0}$ |
| $1 \quad (-\infty < n < \infty)$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$ |
| $a^n u[n] \quad (\|a\| < 1)$ | $\frac{1}{1-ae^{-j\omega}}$ |
| $u[n]$ | $\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$ |
| $(n+1)a^n u[n] \quad (\|a\| < 1)$ | $\frac{1}{(1-ae^{-j\omega})^2}$ |
| $\frac{\sin(\omega_c n)}{\pi n}$ | $X(e^{j\omega}) = \begin{cases} 1 & \|\omega\| < \omega_c \\ 0 & \omega_c < \|\omega\| \le \pi \end{cases}$ |
| $x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$ | $\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$ |
| $e^{j\omega_0 n}$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$ |

# Common z-transform pairs

| Sequence | Transform | ROC |
|---|---|---|
| $\delta[n]$ | $1$ | All $z$ |
| $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\|z\| > 1$ |
| $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\|z\| < 1$ |
| $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 or $\infty$ |
| $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\|z\| > \|a\|$ |
| $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}$ | $\|z\| < \|a\|$ |
| $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| > \|a\|$ |
| $-na^n u[-n-1]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| < \|a\|$ |
| $\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$ | $\frac{1-a^N z^{-N}}{1-az^{-1}}$ | $\|z\| > 0$ |
| $\cos(\omega_0 n)u[n]$ | $\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\|z\| > 1$ |
| $r^n\cos(\omega_0 n)u[n]$ | $\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\|z\| > r$ |