EEE4001F: Digital Signal Processing

Class Test 1

17 March 2011

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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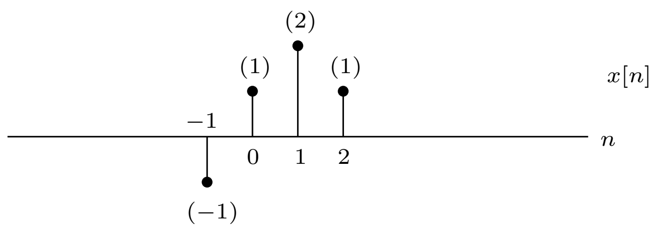
1. (5 marks) If $x[n]$ is the signal below



then plot the following:

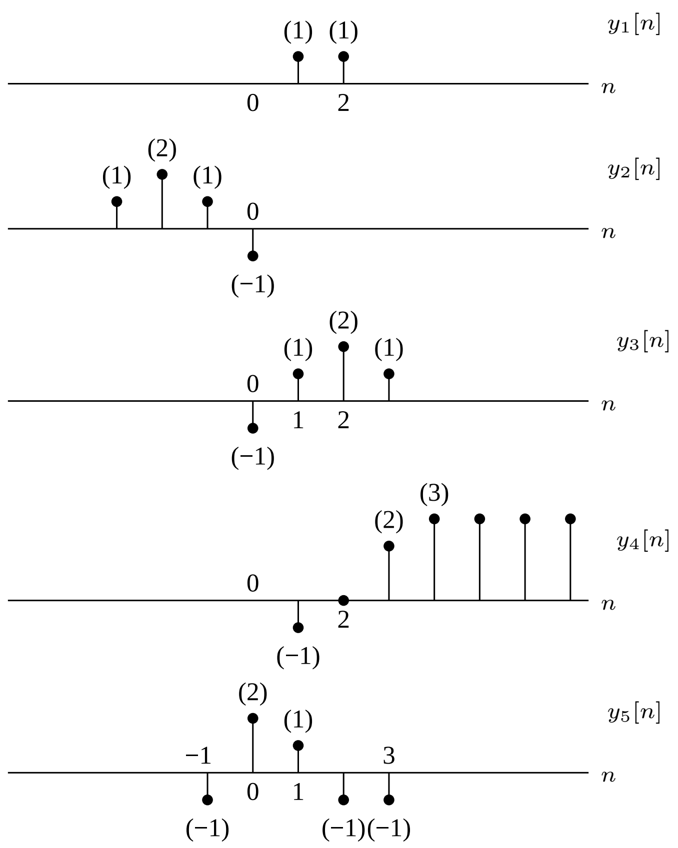
(a) $y_1[n] = x[2n-2]$

(b) $y_2[n] = x[-n-1]$

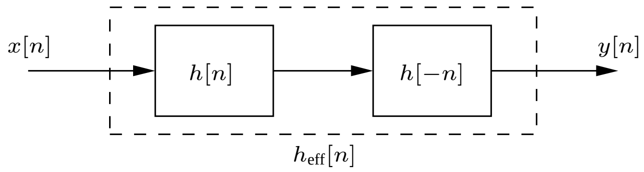
(c) $y_3[n] = x[n] * \delta[n-1]$

(d) $y_4[n] = x[n-1] * u[n-1]$

(e) $y_5[n] = x[n] - x[n-1]$.



2. (5 marks) Consider the system below



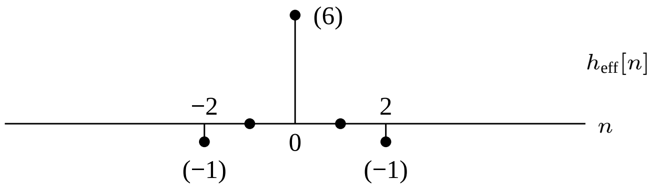
where $h[n] = \delta[n] - 2\delta[n-1] - \delta[n-2]$.

(a) Find $H(e^{j\omega})$, the Fourier transform of $h[n]$.

(b) Find and plot the effective impulse response $h_{\text{eff}}[n]$ linking the input $x[n]$ and the output $y[n]$.

(c) Give an expression for the effective system transfer function $H_{\text{eff}}(e^{j\omega})$ in terms of $H(e^{j\omega})$.

(a) The effective impulse response is $h_{\text{eff}}[n] = h[n] * h[-n]$, shown below:



(b) The Fourier transform is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (\delta[n] - 2\delta[n-1] - \delta[n-2])e^{-j\omega n}$$
$$= 1 - 2e^{-j\omega} - e^{-j2\omega}.$$

(c) The effective Fourier transform is

$$H_{\text{eff}}(e^{j\omega}) = \mathcal{F}\{h[n]\}\mathcal{F}\{h[-n]\} = H(e^{j\omega})H(e^{-j\omega}).$$

Interestingly, but not required for the answer, one can show that this evaluates to a real-valued transform

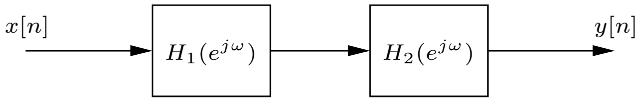
$$H_{\text{eff}}(e^{j\omega}) = 6 - 2\cos(2\omega)$$

for the given $h[n]$. The overall filter is therefore zero phase. In general if $h[n]$ is real then $H(e^{-j\omega}) = H^*(e^{j\omega})$, and

$$H_{\text{eff}}(e^{j\omega}) = H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2$$

will have a zero phase response — which is a special case of linear phase.

3. (5 marks) Suppose we have the following cascade of all-pole filters:



where

$$H_1(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \qquad \text{and} \qquad H_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}}.$$

(a) Find $h[n]$ such that the input $x[n]$ and output $y[n]$ satisfy the relationship $y[n] = h[n] * x[n]$.

(b) Find $g[n]$ such that $x[n] = g[n] * y[n]$.

(a) The given relationship in the frequency domain is $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$. Since $Y(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})X(e^{j\omega})$ we must have

$$\begin{aligned} H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) &= \frac{1}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{3}e^{-j\omega})} \\ &= \frac{3}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{3}e^{-j\omega}}, \end{aligned}$$

and the inverse Fourier transform is the required impulse response

$$h[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n].$$

(b) The given relationship in the frequency domain is $X(e^{j\omega}) = H(e^{j\omega})Y(e^{j\omega})$. Since $Y(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})X(e^{j\omega})$ we must therefore have

$$\begin{aligned} G(e^{j\omega}) = \frac{1}{H_1(e^{j\omega})H_2(e^{j\omega})} &= \left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{3}e^{-j\omega}\right) \\ &= 1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}. \end{aligned}$$

The Fourier inverse is the quantity required in the time domain:

$$g[n] = \delta[n] - \frac{5}{6}\delta[n-1] + \frac{1}{6}\delta[n-2].$$

4. (5 marks) Consider the minimum-phase system with transfer function

$$H(z) = \frac{1 - (1/2)z^{-1}}{1 - (1/3)z^{-1}},$$

where the region of convergence is $|z| > \frac{1}{3}$.

(a) Find a difference equation linking the input $x[n]$ and the output $y[n]$.

(b) Find the impulse response of the system.

(c) Find the transfer function of a causal and stable system that is the inverse of $H(z)$, and sketch the poles, zeros, and region of convergence of this inverse system in the z-plane.

(a) Since $Y(z) = H(z)X(z)$ we have $Y(z)(1 - \frac{1}{3}z^{-1}) = X(z)(1 - \frac{1}{2}z^{-1})$. Inverting gives the difference equation

$$y[n] - \frac{1}{3}y[n-1] = x[n] - \frac{1}{2}x[n-1].$$

(b) The impulse response is the inverse transform of $H(z)$. Since

$$H(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{2}\frac{z^{-1}}{(1 - \frac{1}{3}z^{-1})}$$

the inverse for the given ROC is $h[n] = \left(\frac{1}{3}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{3}\right)^{n-1} u[n-1]$.

(c) The transfer function of the inverse system is

$$G(z) = \frac{1}{H(z)} = \frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

The two possible ROCs are $|z| < 1/2$ and $|z| > 1/2$. For causal and stable the second is appropriate: $|z| > 1/2$. This system has a zero at $z = 1/3$ and a pole at $z = 1/2$, and the sketch follows.

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$