EEE4001F: Digital Signal Processing

Class Test 2

20 April 2012

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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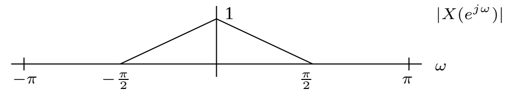
1. (5 marks) An analog signal $x_a(t)$ is known to have no frequency content higher than 1000 Hz. We sample $x_a(t)$ at $F_s = 3000$ Hz, and the resulting magnitude spectrum, plotted versus discrete frequency ω, is

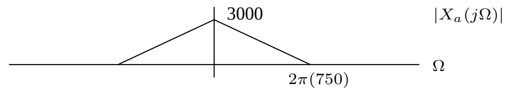


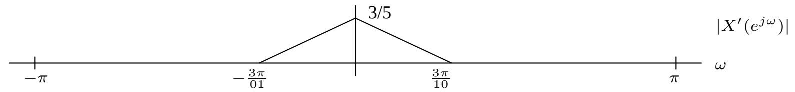
   (a) Sketch the magnitude spectrum (versus discrete frequency ω) that would have resulted had we sampled at $F_s = 5000$ Hz.

   (b) What is highest frequency (in Hz) present in $x_a(t)$?

   (c) What is the lowest sampling frequency that can be used without any aliasing?

   (a) Using the information given we can find the original continuous time signal to have the magnitude spectrum $X_a(j\Omega)$ below, and the corresponding resampled signal has magnitude spectrum $X'(e^{j\omega})$:

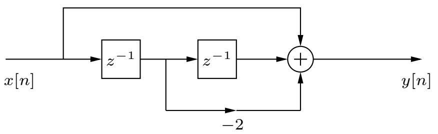




   (b) From the previous solution (or otherwise) the highest frequency present is 750 Hz.

   (c) We need to sample with $F_s \geq 2(750) = 1500$ Hz.

2. (5 marks) Consider the system below, where z^{-1} represents a unit sample delay:



(a) Show that the transfer function is

$$H(z) = 1 - 2z^{-1} + z^{-2}$$

and determine the impulse response.

(b) Sketch the magnitude response and phase response of the filter. Which frequencies are completely blocked?

(a) In the time domain the system obeys the recursion

$$y[n] = x[n] - 2x[n-1] + x[n-2].$$

Taking the z-transform gives $Y(z) = X(z)[1 - 2z^{-1} + z^{-2}]$, so

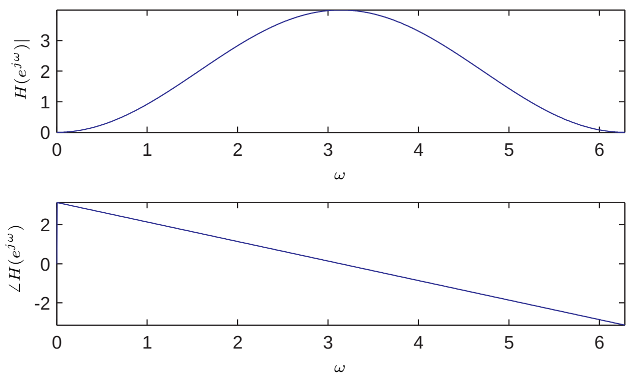
$$H(z) = \frac{Y(z)}{X(z)} = 1 - 2z^{-1} + z^{-2}.$$

Since this is an all-zero filter the ROC is the entire z-plane, and the filter is stable and causal. The impulse response is the inverse: $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$.

(b) Since

$$H(z) = \frac{(z-1)(z-1)}{z^2}$$

the system has two zeros at $z = 1$ and two poles at the origin. Using graphical methods it is easy to see that the magnitude and phase response of the system is as follows:



DC is the only frequency that is completely blocked.

3. (5 marks) A linear time invariant system has system function

$$H(z) = 1 - 2z^{-1} + z^{-2}.$$

Determine the output $y[n]$ when the input is

$$x[n] = 3\cos\left(\frac{\pi}{3}n + \frac{\pi}{6}\right).$$

Write your answer as $y[n] = A\cos(Bn + C)$ for appropriate values of A, B, and C.

Since

$$x[n] = 3/2e^{j(\pi/3n+\pi/6)} + 3/2e^{-j(\pi/3n+\pi/6)} = 3/2e^{j\pi/6}e^{j\pi/3n} + 3/2e^{-j\pi/6}e^{-j\pi/3n}$$

the output is

$$\begin{aligned} y[n] &= 3/2e^{j\pi/6}H(e^{j\pi/3})e^{j\pi/3n} + 3/2e^{-j\pi/6}H(e^{-j\pi/3})e^{-j\pi/3n} \\ &= 3/2H(e^{j\pi/3})e^{j(\pi/3n+\pi/6)} + 3/2H(e^{-j\pi/3})e^{-j(\pi/3n+\pi/6)}. \end{aligned}$$

Since $H(e^{j\pi/3}) = 1e^{j2\pi/3}$ and $H(e^{-j\pi/3}) = 1e^{-j2\pi/3}$ the output can be written as

$$y[n] = 3/2e^{j2\pi/3}e^{j(\pi/3n+\pi/6)} + 3/2e^{-j2\pi/3}e^{-j(\pi/3n+\pi/6)} = 3\cos(\pi/3n + 5\pi/6).$$

Thus $A = 3$, $B = \pi/3$, and $C = 5\pi/6$.

4. (5 marks) The DFT operation can be expressed in the following matrix form:

$$\mathbf{X} = \mathbf{D}_N\mathbf{x},$$

where $\mathbf{X}$ and $\mathbf{x}$ are N-dimensional vectors and $\mathbf{D}_N$ is called the DFT matrix.

(a) Write down in full the matrix $\mathbf{D}_4$ in terms of the quantity $W_4 = e^{-j\frac{2\pi}{4}}$.

(b) Suppose a programming language has a function `fft` such that $\mathbf{X} = \texttt{fft}(\mathbf{x})$. Explain how you could use this function to construct the matrix $\mathbf{D}_N$.

(a) Since

$$X[k] = \sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn} = \sum_{n=0}^{3} x[n]W_4^{kn}$$

we can write

$$\begin{aligned} X[k] &= x[0] + x[1]W_4^k + x[2]W_4^{2k} + x[3]W_4^{3k} \\ &= x[0] + W_4^k x[1] + W_4^{2k}x[2] + W_4^{3k}x[3]. \end{aligned}$$

Thus we can write

$$\mathbf{X} = \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{pmatrix} = \mathbf{D}_4\mathbf{x}.$$

The factors W_4^k in the above matrix $\mathbf{D}_4$ can always be written in an equivalent form with $k \in \{0, 1, 2, 3\}$, although that does somewhat obscure the structure.

(b) If we form the N-dimensional ith unit vector $\hat{\mathbf{e}}_i$ (all elements zero except for a 1 in position i), then $\texttt{fft}(\hat{\mathbf{e}}_i)$ will give the ith column of $\mathbf{D}_N$. Thus we can construct all of $\mathbf{D}_N$ by calling the `fft` function N times with each of the unit vectors.

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$