EEE4001F: Digital Signal Processing

Class Test 1

20 March 2013

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) Given the sequence

$$x[n] = 2\delta[n+3] + (3-n)(u[n] - u[n-3]),$$

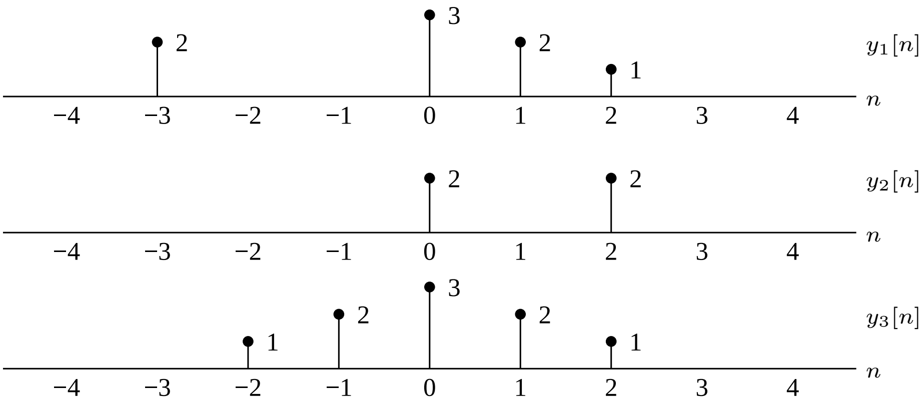
sketch the following sequences (for $-4 \le n \le 4$):

(a) $y_1[n] = x[n]$

(b) $y_2[n] = x[2n-3]$

(c) $y_3[n] = x[|n|]$.

Plots as follows, easily found by just evaluating each signal for each n:



2. (5 marks) A linear time-invariant system has an impulse response given by $h[n] = a^{-n}u[-n]$, $0 < a < 1$, where $u[n]$ is the unit step sequence

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

Determine the response to the input $x[n] = u[n]$.

We want to find the output

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} a^{-k}u[-k]u[n-k]$$

for each value of n. Since $u[-k]$ is zero for $k \geq 1$, and is otherwise 1, the sum can be truncated and the expression written as

$$y[n] = \sum_{k=-\infty}^{0} a^{-k}u[n-k] = \sum_{m=0}^{\infty} a^{m}u[n+m],$$

with $m = -k$. Two cases can now occur. If $n \geq 0$ then $u[n+m]$ in the sum is always one and

$$y[n] = \sum_{m=0}^{\infty} a^m = \frac{1}{1-a}.$$

Alternatively, if $n < 0$ then the first nonzero term in the sum occurs when $m = -n$ so

$$\begin{aligned} y[n] &= \sum_{m=-n}^{\infty} a^m = a^{-n} + a^{-n+1} + a^{-n+2} + \cdots \\ &= a^{-n}(1 + a^{-1} + a^{-2} + \cdots) = \frac{a^{-n}}{1-a}. \end{aligned}$$

The response is therefore

$$y[n] = \begin{cases} a^{-n}\frac{1}{1-a} & n < 0 \\ \frac{1}{1-a} & n \geq 0. \end{cases}$$

3. (5 marks) Consider two discrete-time LTI systems which are characterized by their impulse responses $h_1[n] = \delta[n] - \delta[n-1]$ and $h_2[n] = u[n]$.

   (a) Determine whether these two LTI systems are inverses of each other. Justify your answer.

   (b) Determine whether these systems are stable, memory-less, and causal. Justify your answer.

(a) Suppose $x[n]$ is the input to the first system. The output is then $y_1[n] = h_1[n] * x[n]$. If this signal is put into the second system the output is $z[n] = h_2[n] * y_1[n] = h_2[n] * h_1[n] * x[n]$. However,

$$\begin{aligned} h_1[n] * h_2[n] &= (\delta[n] - \delta[n-1]) * u[n] = \delta[n] * u[n] - \delta[n-1] * u[n] \\ &= u[n] - u[n-1] = \delta[n]. \end{aligned}$$

Thus we see that $z[n] = x[n]$ and the systems are inverses of one another.

(b) The impulse response $h_1[n] = \delta[n] - \delta[n-1]$ corresponds to a backward difference system. It is causal because the output at n only depends on the input at n and $n-1$, none of which are in the future Memory is required to store input at $n-1$. Since $\sum_{n=-\infty}^{\infty} |h_1[n]| = 2 < \infty$ the system is stable. The impulse response $h_2[n] = u[n]$ corresponds to an accumulator system and has the following input-output relationship:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]u[n-k] = \sum_{k=-\infty}^{n} x[k]$$

Since the impulse response is right-sided the system is causal, and because the output at time n depends on all inputs up to n the system requires memory. Furthermore, since the unit step input yields an unbounded output the system is not stable (or alternatively, it is not stable since $\sum_{n=-\infty}^{\infty} |h_2[n]| = \sum_{n=0}^{\infty} 1 \to \infty$).

4. (5 marks) An LTI system is described by the input-output relation

$$y[n] = x[n] + 2x[n-1] + x[n-2].$$

(a) Determine the impulse response $h[n]$

(b) Is this a stable system?

(c) Show that the frequency response of the system can be written as

$$H(e^{j\omega}) = 2e^{-j\omega}(\cos(\omega) + 1).$$

(d) Plot the magnitude and phase of $H(e^{j\omega})$

(e) Now consider a new system whose frequency response is $H_1(e^{j\omega}) = H(e^{j(\omega+\pi)})$. Determine $h_1[n]$, the impulse response of the new system.

(a) The impulse response is the output when the input is $x[n] = \delta[n]$, so

$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$$

(b) The system is stable because the impulse response is absolutely summable:

$$\sum_{n=-\infty}^{\infty} h[n] = 1 + 2 + 1 = 4 < \infty.$$

(c) The z-transform of the system is

$$H(z) = 1 + 2z^{-1} + z^{-2} = (1 + z^{-1})(1 + z^{-1})$$

with ROC all z. Evaluating at $z = e^{j\omega}$ gives the Fourier transform:

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega} = e^{-j\omega}(e^{j\omega} + 2 + e^{-j\omega}) = 2e^{-j\omega}(\cos(\omega) + 1).$$

(d) The magnitude response is $|H(e^{j\omega})| = 2(\cos(\omega) + 1)$ and the phase response is $\angle H(e^{j\omega}) = -\omega$. Plot is a raised cosine with maximum value 4 for the magnitude and a linear function with slope -1 for the phase.

(e) If the Fourier transform of $h[n]$ is $H(e^{j\omega})$ then the frequency shift property is

$$e^{j\omega_0 n}h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega-\omega_0)n} = H(e^{j(\omega-\omega_0)}).$$

Taking $\omega_0 = -\pi$ gives

$$h_1[n] = e^{-j\pi n}h[n] = (-1)^n h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2].$$

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$