EEE4001F: Digital Signal Processing

Class Test 1

11 March 2016

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.
- An information sheet is attached.

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1. (5 marks) A discrete-time system is governed by the following relation:

$$y[n] = \sum_{k=0}^{2} x[n-k] + x[0].$$

   (a) Find the output when the input is $x_1[n] = u[n]$.

   (b) Find the output when the input is $x_2[n] = u[n-1]$.

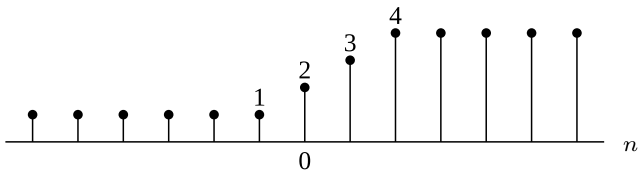
   (c) Is the system time invariant?

The input-output relationship can be written as

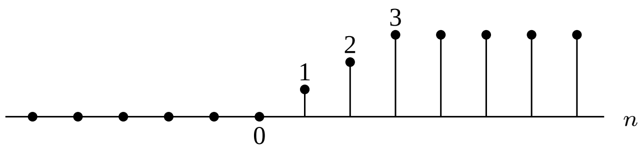
$$y[n] = x[n] + x[n-1] + x[n-2] + x[0],$$

and for each n the required values can be found by direct substitution.

(a) Response for $x_1[n] = u[n]$ is $y_1[n]$ below:



(b) Response for $x_2[n] = u[n-1]$ is $y_2[n]$ below:



(c) Since $x_2[n] = x_1[n]$ but $y_2[n] \neq y_1[n]$ the system is not time invariant.

2. (5 marks) Determine the DTFT of the sequence $x[n] = \alpha^n u[-n-1]$ for $|\alpha| > 1$.

Can approach the problem from first principles:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \alpha^n u[-n-1]e^{-j\omega n} = \sum_{n=-\infty}^{-1} \alpha^n e^{-j\omega n}$$
$$= \sum_{n=1}^{\infty} \alpha^{-n} e^{j\omega n} = -1 + \sum_{n=0}^{\infty} (\alpha^{-1}e^{j\omega})^n.$$

Since we know that $|\alpha| > 1$ we have $|\alpha^{-1}e^{j\omega}| < 1$, so this infinite series converges to

$$X(e^{j\omega}) = -1 + \frac{1}{1-\alpha^{-1}e^{j\omega}} = \frac{\alpha^{-1}e^{j\omega}}{1-\alpha^{-1}e^{j\omega}}$$

Alternatively one could use the given z-transform pair

$$-\alpha^n u[-n-1] \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \frac{1}{1-\alpha z^{-1}} \qquad |z| < |\alpha|$$

to obtain

$$\alpha^n u[-n-1] \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad -\frac{1}{1-\alpha z^{-1}} \qquad |z| < |\alpha|.$$

Thus

$$H(z) = \frac{-1}{1-\alpha z^{-1}} = \frac{z}{\alpha - z} = \frac{\alpha^{-1}z}{1-\alpha^{-1}z} \quad \text{for} \quad |z| < |\alpha|.$$

Now since $|\alpha| > 1$ the ROC includes the unit circle and we can write

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{\alpha^{-1}e^{j\omega}}{1-\alpha^{-1}e^{j\omega}},$$

as before.

3. (5 marks) Suppose a sequence $x[n]$ has DTFT $X(e^{j\omega})$. Find the time-domain inverses of each of the following:

(a) $Y_1(e^{j\omega}) = 2X(e^{-j(\omega-\omega_0)})$, and

(b) $Y_2(e^{j\omega}) = 3e^{j4\omega}X(e^{j(\omega-\omega_0)})$.

Express your answers in terms of $x[n]$.

(a) Applying the time reversal property to the given pair yields

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega}).$$

Applying frequency shifting to this gives the pair

$$e^{j\omega_0 n}x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j(\omega-\omega_0)}).$$

Finally from linearity

$$2e^{j\omega_0 n}x[-n] \overset{\mathcal{F}}{\longleftrightarrow} 2X(e^{-j(\omega-\omega_0)}).$$

Thus $y_1[n] = 2e^{j\omega_0 n}x[-n]$.

(b) Using frequency shifting on the given pair yields

$$e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).$$

Now time shift with $n_d = -4$ on this pair gives

$$e^{j\omega_0(n+4)}x[n+4] \overset{\mathcal{F}}{\longleftrightarrow} e^{j4\omega}X(e^{j(\omega-\omega_0)})$$

Using linearity provides the required result as $y_2[n] = 3e^{j\omega_0(n+4)}x[n+4]$.

4. (5 marks) Consider two discrete-time systems with the following impulse responses:

$$h_1[n] = \delta[n] - \delta[n-1] \qquad \text{and} \qquad h_2[n] = u[n].$$

(a) Are the systems causal? Why?

(b) Using time-domain reasoning show that the systems are inverses of one another.

(c) Draw pole-zero plots of the system functions in each case.

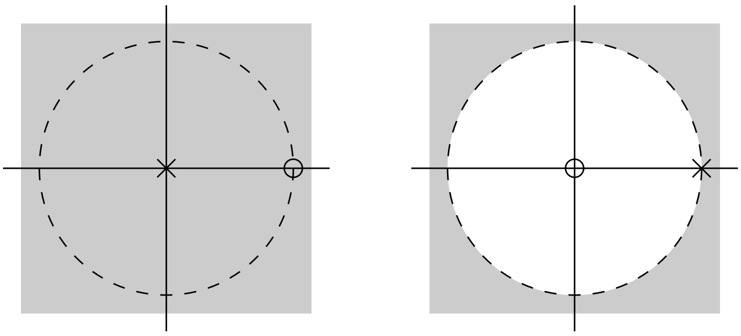
(a) In both cases the impulse response satisfies $h[n] = 0$ for $n < 0$ so the systems are causal.

(b) The combined impulse response will be

$$\begin{aligned} h[n] &= h_1[n] * h_2[n] = (\delta[n] - \delta[n-1]) * u[n] = \delta[n] * u[n] - \delta[n-1] * u[n] \\ &= u[n] - u[n-1] = \delta[n], \end{aligned}$$

so the output of the combined system will be identical to the input. Thus the systems are inverses of one another.

(c) The z-transform of $h_1[n]$ is $H_1(z) = 1 - z^{-1}$ (ROC all z) and the transform of $h_2[n]$ is $H_2(z) = 1/(1 - z^{-1})$ (ROC $|z| > 1$). The pole-zero plots are therefore as follows (left $H_1(z)$ and right $H_2(z)$:



Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$