EEE4001F: Digital Signal Processing

Class Test

13 April 2017

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.
- An information sheet is attached.

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1. (5 marks) The input $x[n]$ and output $y[n]$ of a system are linked by the relation

$$y[n] = T\{x[n]\} = x[-n].$$

Answer the following questions, giving reasons:

(a) Is the system additive?

(b) Is the system homogeneous?

(c) Is the system linear?

(d) Is the system time invariant?

(e) Is the system causal?

Suppose $y_1[n] = T\{x_1[n]\} = x_1[-n]$ and $y_2[n] = T\{x_2[n]\} = x_2[-n]$.

(a) The response to $x[n] = x_1[n] + x_2[n]$ is

$$y[n] = T\{x_1[n] + x_2[n]\} = x_1[-n] + x_2[-n] = y_1[n] + y_2[n],$$

so the system is additive.

(b) The response to $x[n] = ax_1[n]$ is

$$y[n] = T\{ax_1[n]\} = ax_1[-n] = ay_1[n],$$

so the system is homogeneous.

(c) Since it is both additive and homogeneous the system is linear.

(d) The response to $x[n] = x_1[n - p]$ is

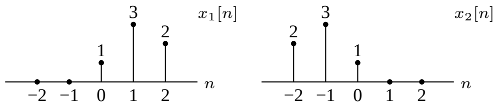
$$y[n] = T\{x_1[n - p]\} = x_1[-(n - p)] \neq y_1[n - p]$$

so the system is not time invariant.

(e) To calculate the output $y[n]$ at time $n = 10$, say, requires knowing the input $x[10]$ which is in the future. Thus the system is not causal.

2. (5 marks) This question deals with convolution and time reversal.

(a) Consider the signals below:

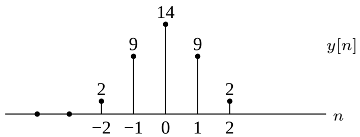


Find $y[n] = x_1[n] * x_2[n]$.

(b) Prove the time reversal property for the z-transform: if $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ then $x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z)$.

(c) Suppose $g[n] = x[n] * x_r[n]$, where $x_r[n] = x[-n]$ is the time reversal of $x[n]$. Show that $g[n]$ is symmetric around the origin, or $g[n] = g[-n]$.

(a) Output is as follows, using the method of your choice:



(b) Suppose $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \sum_{n=\infty}^{\infty} x[n]z^{-n}$. We can derive the time-reversal property:

$$\mathcal{Z}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{m} = X(1/z),$$

so $x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z)$. Thus replacing z with $1/z$ in frequency corresponds to reversing the signal in the time domain.

(c) Since convolution in time is multiplication in frequency we have

$$g[n] = x[n] * x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)X(1/z) = G(z).$$

Observing that $G(z) = G(1/z)$ we can conclude that $g[n] = g[-n]$. Thus convolving a signal with its time reversal always gives a symmetric output.

3. (5 marks) A lowpass filter is described by the following system function:

$$H(z) = \frac{1-a}{1-az^{-1}} \quad \text{with ROC } |z| > |a|.$$

(a) Give an expression for the impulse response of the filter.

(b) Give an expression for the frequency response of the filter. What requirements are there on a for this to exist?

(c) Determine the value of the coefficient a such that the filter has a -3dB cutoff frequency of $\omega = \pi/4$ radians per sample.

(a) Using the transform pair

$$a^n u[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{1}{1-az^{-1}} \quad \text{for } |z| > |a|$$

we obtain the impulse response

$$h[n] = (1-a)a^n u[n].$$

(b) The frequency response is

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{1-a}{1-ae^{-j\omega}}.$$

This requires that the ROC include the unit circle, so $|a| < 1$.

(c) We require that the frequency $\omega = \pi/4$ be at the half-power point, or $|H(e^{j\pi/4})/H(e^{j0})|^2 = 1/2$. Since the DC gain is

$$H(e^{j0}) = \frac{1-a}{1-a} = 1$$

this requires finding a such that $|H(e^{j\pi/4})|^2 = 1/2$. Noting that $e^{j\pi/4} = 1+j$ we get

$$|H(e^{j\pi/4})|^2 = H(e^{j\pi/4})H^*(e^{j\pi/4}) = \frac{1-a}{(1-a(1+j))}\frac{1-a}{(1-a(1-j))}$$

$$= \frac{1-a}{((1-a)-aj)}\frac{1-a}{((1-a)+aj)} = \frac{(1-a)^2}{(1-a)^2+a^2} = \frac{1}{2},$$

so $(1-a)^2 = a^2$ and thus $a = 1/2$.

4. (5 marks) The input to an anticausal LTI system is

$$x[n] = u[-n-1] + (1/2)^n u[n].$$

The Z-transform of the output of the system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}.$$

(a) Determine the system function $H(z)$ and specify the ROC.

(b) Show that the ROC of $Y(z)$ is $1/2 < |z| < 1$ and find the time-domain output $y[n]$.

(a) Here

$$X(z) = -\frac{1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{-\frac{1}{2}z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}$$

with ROC_x as $1/2 < |z| < 1$. Given the output, we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} \frac{(1-z^{-1})(1-\frac{1}{2}z^{-1})}{-\frac{1}{2}z^{-1}} = \frac{1-z^{-1}}{1+z^{-1}}.$$

This has a pole at $z=-1$, so for an anticausal system we require $|z| < 1$ as ROC_h.

(b) We require that ROC_y contain $\mathrm{ROC}_x \cap \mathrm{ROC}_h = 1/2 < |z| < 1$. Now $Y(z)$ has poles at $z = 1/2$ and $z=-1$, so possible ROCs are $|z| < 1/2$, $1/2 < |z| < 1$, and $|z| > 1$. Thus we must have $1/2 < |z| < 1$ for ROC_y. Using partial fractions we can write

$$Y(z) = \frac{1/3}{1+z^{-1}} - \frac{1/3}{1-\frac{1}{2}z^{-1}},$$

and the required inverse is

$$y[n] = -\frac{1}{3}(-1)^n u[-n-1] - \frac{1}{3}(1/2)^n u[n].$$

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n-n_d]$	$e^{-j\omega n_d}X(e^{j\omega})$	Time shift
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$