EEE4001F EXAM
DIGITAL SIGNAL PROCESSING

University of Cape Town
Department of Electrical Engineering

May 2010
3 hours

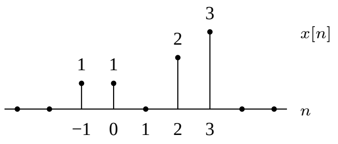
Information

- The exam is closed-book.
- There are two parts to this exam.
- **Part A** has *seven* questions totalling 70 marks. You must answer all of them.
- **Part B** has *two* questions, each counting 15 marks. You must answer *both* of them.
- A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
- A formula sheet for the radar/sonar question appears at the end of this paper.
- You have 3 hours.

PART A

Answer all of the following questions.

1. Consider the signal $x[n]$ below:



Sketch the following:

(a) $y_1[n] = x[-n]$

(b) $y_2[n] = x[2n-1]$

(c) $y_3[n] = x[-2n+1]$

(d) $y_4[n] = x[n] * (\delta[n] - \delta[n-1])$

(e) $y_5[n] = \sum_{k=-\infty}^{n} x[k]$

(f) $y_6[n] = x[n] * u[n]$.

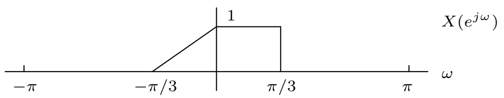
(15 marks)

2. (a) Use the definition of convolution to show that the impulse response $h[n]$ of a causal LTI system must satisfy $h[n] = 0$ for $n < 0$.

(b) What are the potential advantages and disadvantages of a filter with a long impulse response?

(5 marks)

3. A sequence $x[n]$ has a zero-phase DTFT $X(e^{j\omega})$ given below:



(a) Sketch the DTFT of the sequence $y_1[n] = x[n]\cos(-\pi n/3)$.

(b) Sketch the DTFT of the sequence $y_2[n] = x[n] + x[-n]$.

(10 marks)

4. Find $w[n] = x[n] * y[n]$ with

$$x[n] = e^{j\pi n/3} \quad \text{and} \quad y[n] = n(0.7)^n u[n].$$

(10 marks)

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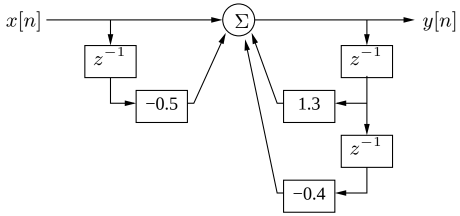
5. Consider the following LTI system:



Determine a closed-form expression for the response $y[n]$ of the system to the unit step input $x[n] = u[n]$. Assume that the system has zero initial conditions. Sketch your result.

(10 marks)

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6. Suppose $x[n]$ is a complex exponential signal with $x[n] = e^{j\frac{2\pi}{12}mn}$ for some integer $0 \le m < 12$.

(a) What is the frequency of the signal $x[n]$?

(b) Let $X[k]$ be the 12-point DFT of $x[n]$. Show that over one interval ($0 \le k < 12$) this signal is $X[k] = \delta[k - m]$:

(c) Use this result to find and plot the 12-point DFT of $y[n] = \cos(\frac{\pi}{3}n)$ over the range $0 \le k < 12$.

(10 marks)

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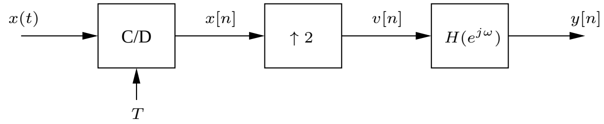
7. Suppose $x(t) = \cos(100\pi t)$ is the input to the system below:



where $T = \frac{1}{400}$s and

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le 0.25\pi \\ 0 & \text{otherwise} \end{cases}$$

for $-\pi \le \omega \le \pi$. Find $Y(e^{j\omega})$ and $y[n]$.

(10 marks)

PART B

Answer *both* of the following two questions. Each question counts 15 marks.

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1. **Image processing and computer vision**

   (a) The human visual system (HVS) has been said to have a bandpass filter response. Present some sources of evidence for this claim.

   (5 marks)

   (b) Explain the sense in which a camera can be modelled as a linear shift-invariant system. Comment particularly on the system properties that make the theory appropriate. What are the PSF and the MTF, and how do they relate to the camera images?

   (5 marks)

   (c) For a specific problem you are given three points of the form (x, y), namely $(1, 2)$, $(2, 2)$, and $(4, 3)$. Your knowledge of the problem suggests that a model of the form $y = c$ is appropriate, where c is an unknown parameter. That is, the points are assumed to ideally lie on a line $y = 0x + c$ for some unknown value of c. Formulate the problem of estimating c as a least-squares problem, and find the best-fit model.

   (5 marks)

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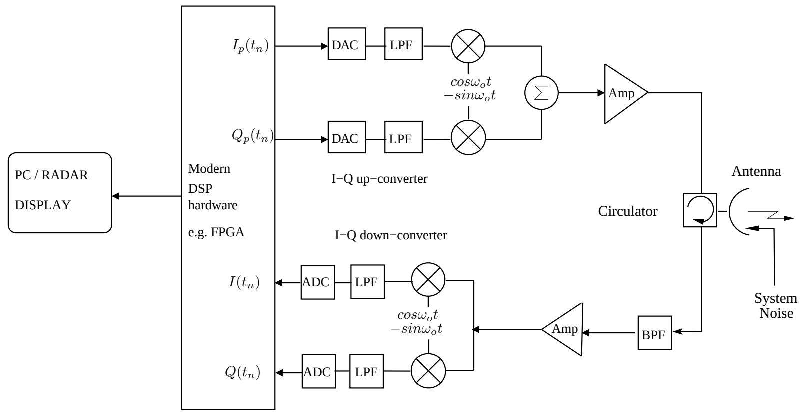
2. **Radar/sonar signal processing**

The block diagram below shows the components of a radar. The radar transmits a chirp pulse of bandwidth 100 MHz, pulse length 100×10^{-6} seconds and centre frequency 10 GHz.



(a) Define what is meant by an “analytic signal” and explain why this representation is useful for analysing the response of a linear system. (3 marks)

(b) Draw an equivalent analytic (complex) signal model of the radar showing the up-conversion and down-conversion processes. (2 marks)

(c) Explain how the signals generated by the DSP hardware (labelled I_p and Q_p) and those sampled by the DSP (labelled I and Q) relate to the transmitted RF waveform and the echo from an arbitrary scene (modelled by its impulse response). As part of your explanation, write down time and frequency domain expressions relating the signals, and include sketches of the frequency spectra of the signals to show their relationship.
(4 marks)

(d) Calculate the range resolution of the radar, assuming that a deconvolution (inverse) filter is used. (1 mark)

(e) The processed radar echo from a point target is sometimes represented as a complex phasor. How is the phase of the phasor related to the target? (1 mark)

(f) A digital signal processing algorithm must be developed for pulse compression and display of the received echoes. The processor should optimize signal to noise ratio *and* keep the side-lobes low. Describe the digital signal processing steps that you would apply to produce a range profile for display from the echo from one transmitted pulse. The inputs to the processor are the sampled I and Q signals. Give details on the filters used, and how you would formulate them. Sketch the expected display output for the case of two point targets in the scene. (4 marks)

(15 marks)

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

FORMULA SHEET V5 EEE4001F 2010

PLEASE REPORT ANY ERRORS TO A.J.W.

Fourier Relationships

$x(t) \leftrightarrow X(f)$

$x(t-t_o) \leftrightarrow X(f)e^{-j2\pi f t_o}$

$x(t)e^{-j2\pi f_o t} \leftrightarrow X(f+f_o)$

$x^*(t) \leftrightarrow X^*(-f)$

$BSa(\pi\beta t) \leftrightarrow rect(\frac{f}{B})$

$rect\left(\frac{t}{\tau}\right) \leftrightarrow \tau Sa(\pi f\tau)$

$\delta(t) \leftrightarrow 1$

For any 'real' signal $x(t)$, $X(-f) = X^*(f)$

Convolution $x(t) \otimes h(t) \leftrightarrow X(f)H(f)$

Radar Equation

$P_r = \frac{P_t G_t \sigma A_e}{(4\pi R^2)^2}$ where $A_e = \frac{G_r \lambda^2}{4\pi}$

IQ Down-converter

$I(t) = [2x(t)\cos(\omega_o t)]_{LPF}$

$Q(t) = [-2x(t)\sin(\omega_o t)]_{LPF}$

$V(t) = I(t) + jQ(t) \leftrightarrow V(f) = 2X^+(f+f_o)$

Matched Filter General

$H(f) = \frac{X^*(f)}{S_{n_i}(f)} \rightarrow X^*(f)$ (white noise)

$\frac{|\upsilon_o(t_{peak})|^2}{\overline{|n_o(t)|^2}} = \frac{E}{\eta/2}$ (white noise)

ANALYTIC RADAR MODEL

Baseband Pulse $p(t)$

Transmitted $\upsilon_{TX}(t) = p(t)e^{j2\pi f_o t}$

EXTENDED TARGET RESPONSE

$\upsilon_{RX}(t) = \int_{\tau=-\infty}^{\infty} \zeta(\tau)\upsilon_{TX}(t-\tau)d\tau = \zeta(t) \otimes \upsilon_{TX}(t)$

$|\zeta(\tau)|^2 \propto \frac{1}{R^4(\tau)} |\beta(\tau)|^2$

$V_{RX}(f) = \zeta(f)V_{TX}(f)$

Baseband Signal

$\upsilon_{bb}(t) = \left[\upsilon_{RX}(t)e^{-j2\pi f_o t}\right] \otimes h_{bb}(t) + n_{bb}(t)$

$\upsilon_{bb}(t) = \left[\zeta(t)e^{-j2\pi f_o t}\right] \otimes p(t) \otimes h_{bb}(t) + n_{bb}(t)$

$V_{bb}(f) = \zeta(f+f_o)\, P(f)\, H_{bb}(f) + N_{bb}(f)$

After Deconvolution/Inverse Filter

$V(f) = \zeta(f+f_o)\mathrm{rect}(\frac{f}{B})$

$\upsilon(t) = \left[\zeta(t)e^{-j2\pi f_o t}\right] \otimes B\frac{\sin(\pi Bt)}{(\pi Bt)}$

where $\frac{\sin(\pi Bt)}{(\pi Bt)} \equiv Sa(\pi Bt)$

POINT TARGET RESPONSE

$\upsilon_{RX}(t) = a_1\upsilon_{TX}(t-\tau)$ where $\tau = \frac{2R}{c}$

$a_1 \propto \sqrt{\frac{G_t G_r \sigma \lambda^2}{(4\pi)^3 R^4}}$ (narrowband)

$\upsilon_{RX}(t) = \sum_{i=1}^{N} a_i \upsilon_{TX}(t-\tau_i)$ where $\tau_i = \frac{2R_i}{c}$

Baseband

$\upsilon_{bb}(t) = \upsilon_{RX}(t)\, e^{-j\omega_o t} \otimes h_{bb}(t) = \zeta\, p(t-\tau)\, e^{-j\omega_o \tau} \otimes h_{bb}(t)$

After deconvolution filtering

$\upsilon(t) = a_1 B\, Sa(\pi B[t-\tau])e^{-j2\pi f_o \tau}$

$\psi = arg\left\{e^{-j2\pi f_o \tau}\right\} = arg\left\{e^{-j4\pi R/\lambda}\right\}$

Resolution

$\delta t_{3dB} \approx \frac{0.89}{B}$ $\delta R_{3dB} = \frac{c\delta t_{3dB}}{2} \approx \frac{c}{2B}(0.89)$

Radar Filters

Ideal Spectral Reconstruction (deconvolution/inverse) Filter

$H_{IRF}(f) = \frac{1}{P(f)H_{bb}(f)}$ over $-\frac{B}{2} \le f \le \frac{B}{2}$

Matched Filter (MF) $H_{MF}(f) = \frac{P^*(f)}{H_{bb}(f)} \approx P^*(f)$

Doppler Shift $f_D = \frac{-2\, dR/dt}{\lambda}$

MONOCHROME PULSE

RF: $\upsilon_{RF}(t) = rect\left(\frac{t}{T}\right)\cos(2\pi f_o t)$

Analytic: $\upsilon_{TX}(t) = rect\left(\frac{t}{T}\right)e^{j2\pi f_o t}$

Baseband: $\upsilon_{bb}(t) = rect\left(\frac{t}{T}\right)$

Frequency Domain

$V_{TX}(f) = T\frac{\sin(\pi T(f-f_o))}{\pi T(f-f_o)}$

$V_{bb}(f) = T\frac{\sin(\pi Tf)}{(\pi Tf)}$

LINEAR FM CHIRP

RF signal $\upsilon_{RF}(t) = rect\left(\frac{t}{T}\right)\cos\left(2\pi[f_o t + \frac{1}{2}Kt^2]\right)$

Analytic: $\upsilon_{TX}(t) = rect\left(\frac{t}{T}\right)e^{j2\pi[f_o t + \frac{1}{2}Kt^2]}$

Baseband: $\upsilon_{bb}(t) = rect\left(\frac{t}{T}\right)e^{j2\pi\frac{1}{2}Kt^2}$

Sweep range $\Delta f = KT$ [Hz]

Instantaneous Frequency

RF: $f_{RF}(t) = \frac{1}{2\pi}\frac{d\psi_{RF}(t)}{dt} = f_o + Kt$ $[Hz]$

Baseband: $f_{bb}(t) = Kt$ [Hz]

Dispersion factor $D = \Delta f\, T = KT^2$

Frequency Domain D¿50

$|\upsilon_{bb}(f)| \approx rect\left(\frac{f}{\Delta f}\right)\frac{1}{\sqrt{|K|}}$

$arg\{\upsilon_{bb}(f)\} = W\left\{-j\frac{\pi}{K}f^2\right\}$