EEE4001F EXAM
DIGITAL SIGNAL PROCESSING

University of Cape Town
Department of Electrical Engineering

June 2012
3 hours

Information

- The exam is closed-book.

- There are two parts to this exam.

- **Part A** has *seven* questions totalling 70 marks. You must answer all of them.

- **Part B** has *two* questions, each counting 15 marks. You must answer *both* of them.

- A table of standard Fourier transform and z-transform pairs appears at the end of this paper.

- A formula sheet for the radar/sonar question appears at the end of this paper.

- You have 3 hours.

PART A

Answer all of the following questions.

---

1. An LTI system has impulse response $h[n] = 5(1/2)^n u[n]$.

   (a) Is the system stable in the bounded-input bounded-output sense? Why?

   (b) Is the system causal? Why?

   (c) Use the discrete-time Fourier transform to find the output of this system $y[n]$ when the input is $x[n] = (1/3)^n u[n]$.

   (10 marks)

---

2. The continuous-time signal $x(t) = \sin(20\pi t) + \cos(40\pi t)$ is sampled with a sampling period T, with $t = nT$, to obtain the discrete-time signal

$$x[n] = \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right).$$

   (a) Sketch the spectrum $X(e^{j\omega})$ of $x[n]$.

   (b) Determine a value of T consistent with this information.

   (c) Is your choice of T in this last part unique? If so, explain why. If not, specify another choice of T consistent with the information given.
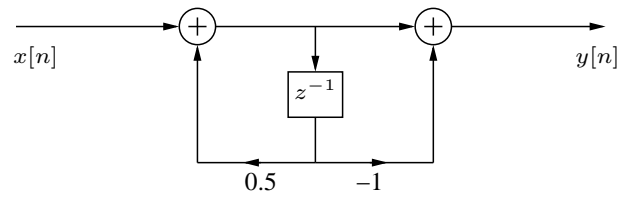
   (10 marks)

---

3. You have a digital filter with zeros at $z = 1$ and $z = -1$, with poles at $z = 0.9e^{j\pi/2}$ and $z = 0.9e^{-j\pi/2}$.

   (a) Sketch the magnitude response for $0 \le \omega \le 2\pi$.

   (b) Find the gain of the system for $\omega = \pi/2$.

   (c) Is this filter lowpass, bandpass or highpass?

   (d) Draw an implementation diagram for this filter using delay elements, multipliers, and summers.

   (10 marks)

---

4. Consider the system below, where z^{-1} represents a delay of one sample:

(a) Determine the system function $H(z)$ and show that the transfer function is

$$H(e^{j\omega}) = \frac{1 - e^{-j\omega}}{1 - 0.5e^{-j\omega}}$$

assuming it is stable. Justify all reasoning.

(b) Find the output $y[n]$ when the input is

$$x[n] = 3\cos\left(\frac{\pi}{3}n + \frac{\pi}{6}\right).$$

(c) Find the output $y[n]$ when the input is $x[n] = 2u[n]$.

(10 marks)

5. When the input to an LTI system is $x[n] = 5u[n]$, the output is

$$y[n] = (2(1/2)^n + 3(-3/4)^n)u[n].$$

(a) Show that the system function can be written as

$$H(z) = \frac{1 - z^{-1}}{(1 - 1/2z^{-1})(1 + 3/4z^{-1})}.$$

(b) Plot the poles and zeros of the system, and indicate the ROC.

(c) Determine the impulse response $h[n]$ of the system for all n.

(d) Write a difference equation that characterises the system.
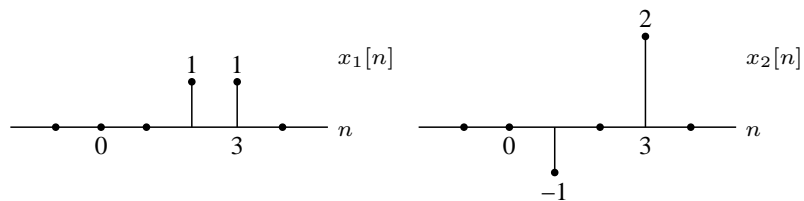
(10 marks)

6. Computationally efficient algorithms for computing the DFT normally exploit the following two properties:

$$(\text{Periodicity}): \quad W_N^{k+N} = W_N^k$$

$$(\text{Symmetry}): \quad W_N^{k+N/2} = -W_N^k,$$

where $W_N = e^{-j2\pi/N}$.

(a) What does W represent in this context, and how does it relate to the DFT?

(b) Prove that the two properties above are true.

(c) Suppose we take the 4-point FFTs of the signals below, multiply them in frequency, and take the inverse FFT of the result:

Sketch the final output over the range $-5 \le n \le 5$.

(10 marks)

7. Consider an input signal $x[n]$ with magnitude spectrum shown below:
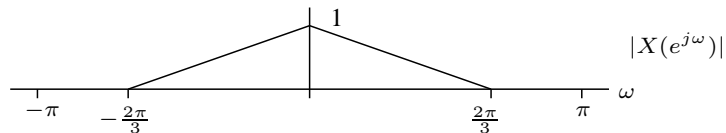
Draw a block diagram of a multirate signal processing system containing multirate processing and a single filter $H(z)$ to reduce the sampling rate of $x[n]$ by a factor of $M = 1.5$. Sketch the magnitude spectra of all intermediate signals and the output signal.
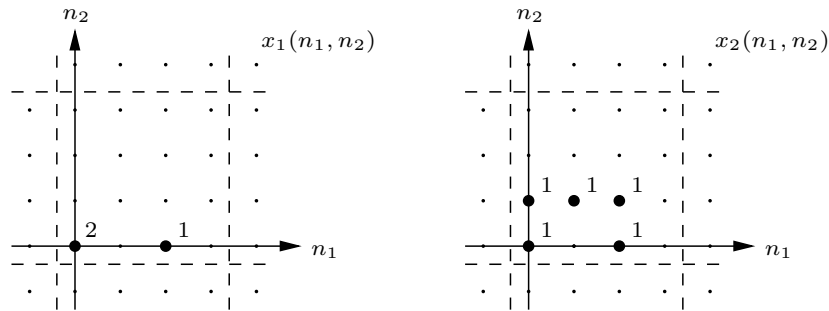
(10 marks)

PART B

Answer *both* of the following two questions. Each question counts 15 marks.

1. **Image processing and computer vision**

   (a) The (N_1, N_2)-point circular convolution between $x_1(n_1, n_2)$ and $x_2(n_1, n_2)$ is defined as
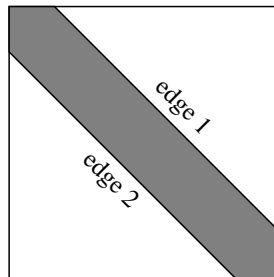
   $$y(n_1, n_2) = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} x_1(k_1, k_2) x_2(((n_1 - k_1))_{N_1}, ((n_2 - k_2))_{N_2})$$

   where $((k))_N$ represents k modulo N. Find $y(n_1, n_2)$ for the two signals below, where $N_1 = N_2 = 4$:

   (5 marks)

   (b) Suppose you need to detect the diagonal edges in the image below, and wish to do it by convolving with a particular kernel and thresholding the result:

   What kernel should you use? What is the difference between the output from the top diagonal edge (edge 1) and the output from the bottom (edge 2), and what would you do if you wanted to detect either (or both) of them? How could noise reduction be incorporated into the detection process?

   (5 marks)

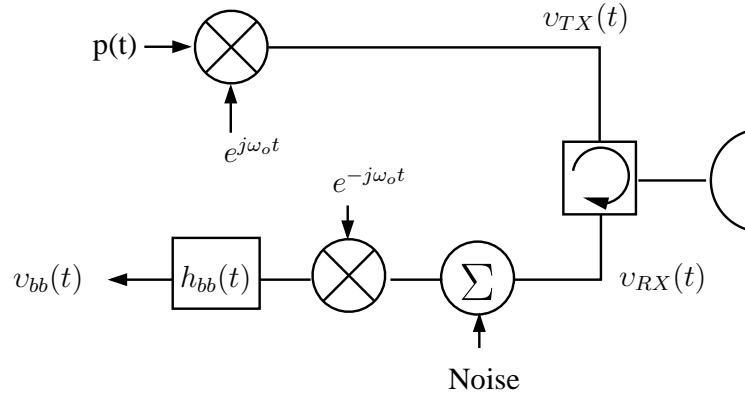(c) The two-dimensional DFT of a signal $x(n_1, n_2)$ is given by

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-j(2\pi/N_1)k_1 n_1 - j(2\pi/N_2)k_2 n_2}.$$

Suppose you wanted to find the 2D DFT of an image. Explain, with the aid of mathematics, how you would use a 1D FFT routine to do this.

(5 marks)

2. Radar/sonar signal processing

An *analytic* signal model for a pulsed radar is shown below.

(a) Draw a block diagram of a physical implementation of this radar showing an I/Q up-converter, an I-Q down-converter, and appropriate sampling into a digital signal processing computer.

(2 marks)

(b) The transmitted waveform can be represented mathematically in three ways: (i) a real RF pulse, (ii) the analytic signal form, and (iii) the basebanded analytic form. Define the relationship between these three forms of signal, in both time and frequency domains. Give appropriate mathematical descriptions of the relationships.

(3 marks)

(c) For the case of a RF chirp pulse: $v_{RF}(t) = \text{rect}\left(\frac{t}{T}\right) \cos\left(2\pi[f_o t + \frac{1}{2}Kt^2]\right)$ where K, f_o and T are constants, sketch the magnitude and phase of the *Fourier Transform* of (i) the RF pulse, (ii) its analytic form, and (iii) its basebanded form.

(3 marks)

(d) What properties of the transmitted waveform determine (i) the range resolution of a radar and (ii) the signal-to-noise ratio?

(2 marks)

(e) State the name and give a mathematical expression for the filter that achieves maximum peak signal to noise ratio if the received echo is from a small scatterer, and the system noise is white noise.

(1 mark)

(f) Explain the operation of a "spectral reconstruction filter" (inverse filter). What is it designed to do, and what will its output be if applied to the basebanded analytic signal (for an arbitrary scene in view of the radar)?

If the scene contains a single point scatterer at range R metres, with impulse response $\zeta(t) = a_1(t - 2R/c)$, where a_1 and c are positive constants, sketch the expected output of the filter as a function of time (show both magnitude and phase).

(4 marks)

Fourier transform properties

Sequences $x[n], y[n]$	Transforms $X(e^{j\omega}), Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n + 1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \|\omega\| < \omega_c \\ 0 & \omega_c < \|\omega\| \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$
$-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$\|z\| < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N - 1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$\|z\| > r$

FORMULA SHEET V6 EEE4001F   2012

PLEASE REPORT ANY ERRORS TO A.J.W.

Fourier Relationships

$x(t) \leftrightarrow X(f)$

$x(t - t_o) \leftrightarrow X(f)e^{-j2\pi f t_o}$

$x(t)e^{-j2\pi f_o t} \leftrightarrow X(f + f_o)$

$x^*(t) \leftrightarrow X^*(-f)$

$BSa(\pi\beta t) \leftrightarrow rect(\frac{f}{B})$

$rect\left(\frac{t}{\tau}\right) \leftrightarrow \tau Sa(\pi f \tau)$

$\delta(t) \leftrightarrow 1$

For any 'real' signal $x(t)$, $X(-f) = X^*(f)$

Convolution $x(t) \circledast h(t) \leftrightarrow X(f)H(f)$

Radar Equation

$P_r = \frac{P_t G_t \sigma A_e}{(4\pi R^2)^2}$ where $A_e = \frac{G_r \lambda^2}{4\pi}$

IQ Down-converter

$I(t) = [2x(t)\cos(\omega_o t)]_{LPF}$

$Q(t) = [-2x(t)\sin(\omega_o t)]_{LPF}$

$V(t) = I(t) + jQ(t) \leftrightarrow V(f) = 2X^+(f + f_o)$

Matched Filter General

$H(f) = \frac{X^*(f)}{S_{n_i}(f)} \rightarrow X^*(f)$ (white noise)

$\frac{|v_o(t_{peak})|^2}{|n_o(t)|^2} = \frac{E}{\eta/2}$ (white noise)

ANALYTIC RADAR MODEL

Baseband Pulse $p(t)$

Transmitted $v_{TX}(t) = p(t)e^{j2\pi f_o t}$

EXTENDED TARGET RESPONSE

$v_{RX}(t) = \int_{\tau=-\infty}^{\infty} \zeta(\tau)v_{TX}(t - \tau)d\tau = \zeta(t) \circledast v_{TX}(t)$

$|\zeta(\tau)|^2 \propto \frac{1}{R^4(\tau)}|\beta(\tau)|^2$

$V_{RX}(f) = \zeta(f)V_{TX}(f)$

Baseband Signal

$v_{bb}(t) = \left[v_{RX}(t)e^{-j2\pi f_o t}\right] \circledast h_{bb}(t) + n_{bb}(t)$

$v_{bb}(t) = \left[\zeta(t)e^{-j2\pi f_o t}\right] \circledast p(t) \circledast h_{bb}(t) + n_{bb}(t)$

$V_{bb}(f) = \zeta(f + f_o)P(f)H_{bb}(f) + N_{bb}(f)$

After Deconvolution/Inverse Filter

$V(f) = \zeta(f + f_o)rect(\frac{f}{B})$

$v(t) = \left[\zeta(t)e^{-j2\pi f_o t}\right] \circledast B\frac{\sin(\pi B t)}{(\pi B t)}$

where $\frac{\sin(\pi B t)}{(\pi B t)} \equiv Sa(\pi B t)$

POINT TARGET RESPONSE

$v_{RX}(t) = a_1 v_{TX}(t - \tau)$ where $\tau = \frac{2R}{c}$

$a_1 \propto \sqrt{\frac{G_t G_r \sigma \lambda^2}{(4\pi)^3 R^4}}$ (narrowband)

$v_{RX}(t) = \sum_{i=1}^{N} a_i v_{TX}(t - \tau_i)$ where $\tau_i = \frac{2R_i}{c}$

Baseband

$v_{bb}(t) = v_{RX}(t)\, e^{-j\omega_o t} \circledast h_{bb}(t) = \zeta\, p(t - \tau)\, e^{-j\omega_o \tau} \circledast h_{bb}(t)$

After deconvolution filtering

$v(t) = a_1 B\, Sa(\pi B[t - \tau])e^{-j2\pi f_o \tau}$

$\psi = arg\left\{ e^{-j2\pi f_o \tau} \right\} = arg\left\{ e^{-j4\pi R/\lambda} \right\}$

Resolution

$\delta t_{3dB} \approx \frac{0.89}{B}$ $\delta R_{3dB} = \frac{c\delta t_{3dB}}{2} \approx \frac{c}{2B}(0.89)$

Radar Filters

Ideal Spectral Reconstruction (deconvolution/inverse) Filter

$H_{\text{IRF}}(f) = \frac{1}{P(f)H_{bb}(f)}$ over $-\frac{B}{2} \leq f \leq \frac{B}{2}$

Matched Filter (MF) $H_{\text{MF}}(f) = \frac{P^*(f)}{H_{bb}(f)} \approx P^*(f)$

Doppler Shift $f_D = \frac{-2\, dR/dt}{\lambda}$

MONOCHROME PULSE

RF: $v_{RF}(t) = rect\left(\frac{t}{T}\right) \cos(2\pi f_o t)$

Analytic: $v_{TX}(t) = rect\left(\frac{t}{T}\right) e^{j2\pi f_o t}$

Baseband: $v_{bb}(t) = rect\left(\frac{t}{T}\right)$

Frequency Domain

$V_{TX}(f) = T\frac{\sin(\pi T(f - f_o))}{\pi T(f - f_o)}$

$V_{bb}(f) = T\frac{\sin(\pi T f)}{(\pi T f)}$

LINEAR FM CHIRP

RF signal $v_{RF}(t) = rect\left(\frac{t}{T}\right) \cos\left(2\pi[f_o t + \frac{1}{2}Kt^2]\right)$

Analytic: $v_{TX}(t) = rect\left(\frac{t}{T}\right) e^{j2\pi[f_o t + \frac{1}{2}Kt^2]}$

Baseband: $v_{bb}(t) = rect\left(\frac{t}{T}\right) e^{j2\pi\frac{1}{2}Kt^2}$

Sweep range $\Delta f = KT$ [Hz]

Instantaneous Frequency

RF: $f_{RF}(t) = \frac{1}{2\pi}\frac{d\psi_{RF}(t)}{dt} = f_o + Kt$ [Hz]

Baseband: $f_{bb}(t) = Kt$ [Hz]

Dispersion factor $D = \Delta f\, T = KT^2$

Frequency Domain $D > 50$

$V_{bb}(f) \approx rect\left(\frac{f}{\Delta f}\right) \frac{1}{\sqrt{|K|}}\exp\left(-j\frac{\pi}{K}f^2\right)$