# EEE4001F EXAM
# DIGITAL SIGNAL PROCESSING

University of Cape Town
Department of Electrical Engineering

June 2015
3 hours

---

**Information**

- The exam is closed-book.
- There are two parts to this exam.
- **Part A** has *six* questions totalling 50 marks. You must answer all of them.
- **Part B** has *ten* questions totalling 50 marks. You must answer all of them.
- Parts A and B must be answered in different sets of exam books, which will be collected separately.
- A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
- You have 3 hours.

---

# PART A

Digital signal processing.

---

1. Consider a causal linear time-invariant system which results in the output

$$y[n] = \left(\frac{1}{3}\right)^n u[n]$$

when the input is

$$x[n] = \frac{1}{4}\left(\frac{1}{2}\right)^{n+1} u[n+1].$$

(a) Plot $x[n]$.

(b) Determine a closed-form expression for the impulse response $h[n]$ of the system.

(c) Is the system stable? Why?

(10 marks)

---

2. (a) An LTI system has impulse response $h[n] = 5(1/2)^n u[n]$ where $u[n]$ is the unit step sequence. Use the discrete-time Fourier transform to find the output of this system when the input is $x[n] = (1/3)^n u[n]$.

(b) Find the signal $h[n]$ with the following DTFT:

$$H(e^{j\omega}) = 2(e^{j\omega})^2 - \frac{3(e^{j\omega})^{-3}}{e^{j\omega} - \frac{1}{2}}.$$

(10 marks)

---

3. Let $x[n]$ be the input and $y[n]$ the output of a finite impulse response filter such that

$$y[n] = 4x[n] - x[n-2].$$

(a) Find the poles and zeros of the filter and plot them in the z-plane.

(b) Sketch the magnitude of the frequency response.

(c) Determine the gain of the filter at frequencies 0 and $\pi/2$ radians per sample.

(d) Find an expression for the magnitude of the frequency response of this filter.

(10 marks)

---

4. (a) You measure

$$X_1[0] = 4, \quad X_1[1] = -j4, \quad X_1[2] = -2, \quad X_1[3] = j4$$

and

$$X_2[0] = 2, \quad X_2[1] = 1 - j, \quad X_2[2] = 2, \quad X_2[3] = 1 + j,$$

where $X_1[k]$ is the DFT of $x_1[n]$ and $X_2[k]$ is the DFT of $x_2[n]$. What is the 4-point circular convolution of $x_1[n]$ and $x_2[n]$?

(b) How you would use the FFT to do linear convolution of two signals, each of length 4?

(10 marks)

---

5. Specify a scheme for reducing the sampling rate of a signal to 0.75 of its original sampling frequency. Sketch the magnitude of the frequency response of any filters employed.

(5 marks)

---

6. Suppose a linear time-invariant system is described by the following system function:

$$H(z) = \frac{(z - \frac{1}{2})(z + 2)(z^2 + \frac{1}{9})}{(z^2 + 2z + 5)(z^2 - 4z + 13)}.$$

(a) Draw a pole-zero plot for the system

(b) Determine all possible regions of convergence, and for each indicate whether the corresponding inverse z-transform is left-sided, right-sided, or two-sided. What can you say about the stability of $H(z)$?

(5 marks)

---

# PART B

Wavelets and frames.

---

**P1:** Let the function $f(x)$ be defined by

$$f(x) = e^{-\frac{1}{2}x^2} \qquad -\infty < x < \infty \tag{1}$$

**P1-a:** Calculate $\|f(x)\|_2$, the $L_2-$norm of $f(x)$.

**P1-b:** Let $\widetilde{f}(x)$ denote $f(x)$ normalized. Write down the expression for $\widetilde{f}(x)$.

(2 + 1 = 3 Marks)

---

**P2:** Consider the following complete set of orthogonal functions on the interval $(-\pi, \pi)$:

$$1, \{\cos(nt)\,|n \in \mathbb{N}\}, \{\sin(nt)\,|n \in \mathbb{N}\} \tag{2}$$

**P2-a:** Calculate the $L_2-$norm of the functions $1$, $\cos(nt)$ and $\sin(nt)$ on the interval $(-\pi, \pi)$.

**P2-b:** Utilizing Dirac's bra-ket notation, consider the following resolution of identity for the $L_2-$space of functions with support $(-\pi, \pi)$:

$$\begin{aligned}\mathbb{I} = \quad & |\frac{1}{\sqrt{2\pi}} >< \frac{1}{\sqrt{2\pi}}| \\ + \quad & \sum_{n\in\mathbb{N}} |\frac{1}{\sqrt{\pi}}\cos(nt) >< \frac{1}{\sqrt{\pi}}\cos(nt)\,| \\ + \quad & \sum_{n\in\mathbb{N}} |\frac{1}{\sqrt{\pi}}\sin(nt) >< \frac{1}{\sqrt{\pi}}\sin(nt)\,| \end{aligned} \tag{3}$$

Let the function $f(t)$ satisfy Dirichlet's conditions on the interval $(-\pi, \pi)$, and be zero outside this interval. In bra-ket notation write $|f(t) >$ for $f(t)$. Assume that the operation of (3) from the left onto $|f(t) >$ results in:

$$\begin{aligned}\mathbb{I}|f(t) >= \quad & |\frac{1}{\sqrt{2\pi}} >< \frac{1}{\sqrt{2\pi}}|f(t) > \\ + \quad & \sum_{n\in\mathbb{N}} |\frac{1}{\sqrt{\pi}}\cos(nt) >< \frac{1}{\sqrt{\pi}}\cos(nt)\,|f(t) > \end{aligned} \tag{4}$$

Is $f(t)$ an even function or an odd function?

**P2-c:** Let the function $f(t)$ satisfy Dirichlet's conditions on the interval $(-\pi, \pi)$, and be zero outside this interval. In bra-ket notation write $|f(t) >$ for $f(t)$. Assume that the operation of (3) from the left onto $|f(t)>$ results in:

$$\mathbb{I}|f(t)>=\sum_{n\in\mathbb{N}}|\frac{1}{\sqrt{\pi}}\sin(nt)><\frac{1}{\sqrt{\pi}}\sin(nt)|f(t)> \tag{5}$$

Is $f(t)$ an even function or an odd function?

**P2-d:** What is the expression for the resolution of identity, which characterizes the space of **even** functions with support $(-\pi,\pi)$.

**P2-e:** What is the expression for the resolution of identity, which characterizes the space of **odd** functions with support $(-\pi,\pi)$?

$(2+1+1+2+2=8$ Marks)

---

**P3:** Let the **normalized** functions $\varphi(t)$ and $\psi(t)$ denote the scaling function and the wavelet of a Multiresolution Analysis (MRA) in Hilbert space, respectively. Let the function $\varphi(t)$ generate the function space $\nu_0$. Let the function $\psi(t)$ and its compressed versions generate the spaces , $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\cdots$. Consider the following representation for $f(t)$:

$$f(t)=\sum_{k=-\infty}^{\infty}c_k\varphi(t-k)+\sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty}d_{j,k}2^{\frac{j}{2}}\psi(2^jt-k) \tag{6}$$

**P3-a:** Write down the expression for $c_k$.

**P3-b:** Write down the expression for $d_{j,k}$.

**P3-c:** Write down the expression for the resolution of identity using the set of scaling and wavelet functions!

$(2+2+2=6$ Marks)

---

**P4-a:** Given general ''low pass'' filter coefficient $h(n)$ write down the two-scale dilation equation for the normalized scaling function $\varphi(t)$.

**P4-b:** Given general ''high pass'' filter coefficients $g(n)$ write down the two-scale dilation equation for the normalized wavelet $\psi(t)$.

$(2+2=4$ Marks)

---

**P5-a:** Determine the filter coefficients $h(n)$ for the triangle (piece-wise linear) scaling function.

**P5-b:** Do the coefficients $h(n)$ constitute a ''low pass'' filter or a ''high pass'' filter? Why?

**P5-c:** Determine the filter coefficients $g(n)$ for the triangle (piece-wise linear) wavelet.

**P5-d:** Do the coefficients $g(n)$ constitute a ''low pass'' filter or a ''high pass'' filter? Why?

(2 + 2 + 2 + 2 = 8 Marks)

---

**P6:** Consider the fairly general function $f(t)$. Denote the Fourier transform of $f(t)$ by $F(\omega)$. Construct $F_{\mathcal{M}}(\omega)$ as follows:

$$F_{\mathcal{M}}(\omega) = \frac{F(\omega)}{\sqrt{\sum_{n=-\infty}^{\infty} |F(\omega + 2\pi n)|^2}} \tag{7}$$

Let $f_{\mathcal{M}}(t)$ denote the inverse Fourier transform of $F_{\mathcal{M}}(\omega)$. Using the Parseval's Theorem calculate the norm of $f_{\mathcal{M}}(t)$.

(4 Marks)

---

**P7:** Using the general dilation equation for the wavelet function, express the five-generations compressed normalized wavelet

$$2^{\frac{5}{2}}\psi(2^5 t - m)$$

in terms of linear superposition of $2^{\frac{6}{2}}\varphi(2^6 t - n)$ over $n$.

(4 Marks)

---

**P8:** Construct and plot the Mexican-hat wavelet $\mathcal{M}_h(t)$.

(1 + 1 = 2 Marks)

---

**P9:** Let $|\mathbf{e}_1>$ and $|\mathbf{e}_2>$ be unit normal vectors in the $(x, y)-$plane.

Let the vectors $|\mathbf{f}_1>$ and $|\mathbf{f}_2>$ be defined by the following equations:

$$|\mathbf{f}_1> \;=\; 4|\mathbf{e}_1> -|\mathbf{e}_2> \tag{8a}$$

$$|\mathbf{f}_2> \;=\; 3|\mathbf{e}_1> +2|\mathbf{e}_2> \tag{8b}$$

**P9-a:** Construct the dual vectors $<\widetilde{\mathbf{f}}_1|$ and $<\widetilde{\mathbf{f}}_2|$ corresponding to $|\mathbf{f}_1>$ and $|\mathbf{f}_2>$, respectively, first graphically and then analytically.

**P9-b:** Employ Dirac's bra-ket notation. Resolve the identity operator $\mathbb{I}$ in terms of the ket-vectors $|\mathbf{f}_1>$ and $|\mathbf{f}_2>$ and their dual bra-vectors $<\widetilde{\mathbf{f}}_1|$ and $<\widetilde{\mathbf{f}}_2|$.

(2 + 1 = 3 Marks)

---

**P10:** Let $|\mathbf{e}_1>$ and $|\mathbf{e}_2>$ denote unit normal vectors in the $(x, y)-$plane.

Let the ket vectors $|\mathbf{f}_1>$, $|\mathbf{f}_2>$ and $|\mathbf{f}_3>$ be defined by the following equations:

$$|\mathbf{f}_1> \;=\; 2|\mathbf{e}_1> -|\mathbf{e}_1> \tag{9a}$$

$$|\mathbf{f}_2> \;=\; 3|\mathbf{e}_1> +|\mathbf{e}_2> \tag{9b}$$

$$|\mathbf{f}_3> \;=\; 2|\mathbf{e}_1> +2|\mathbf{e}_2> \tag{9c}$$

The over-complete set of vectors $|\mathbf{f}_1>$, $|\mathbf{f}_2>$ and $|\mathbf{f}_3>$ constitutes a frame.

**P10-a:** Determine the dual frame (bra vectors) $<\widetilde{\mathbf{f}}_1|$, $<\widetilde{\mathbf{f}}_2|$ and $<\widetilde{\mathbf{f}}_3|$.

**P10-b:** Resolve the identity operator $\mathbb{I}$ in the plane in terms of the frame vectors $|\mathbf{f}_1>$, $|\mathbf{f}_2>$ and $|\mathbf{f}_3>$ and their corresponding dual frame vectors $<\widetilde{\mathbf{f}}_1|$, $<\widetilde{\mathbf{f}}_2|$ and $<\widetilde{\mathbf{f}}_3|$.

(4 + 4 = 8 Marks)

---

# Fourier transform properties

| Sequences $x[n], y[n]$ | Transforms $X(e^{j\omega}), Y(e^{j\omega})$ | Property |
|---|---|---|
| $ax[n] + by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ | Linearity |
| $x[n - n_d]$ | $e^{-j\omega n_d} X(e^{j\omega})$ | Time shift |
| $e^{j\omega_0 n} x[n]$ | $X(e^{j(\omega - \omega_0)})$ | Frequency shift |
| $x[-n]$ | $X(e^{-j\omega})$ | Time reversal |
| $nx[n]$ | $j \frac{dX(e^{j\omega})}{d\omega}$ | Frequency diff. |
| $x[n] * y[n]$ | $X(e^{-j\omega})Y(e^{-j\omega})$ | Convolution |
| $x[n]y[n]$ | $\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ | Modulation |

# Common Fourier transform pairs

| Sequence | Fourier transform |
|---|---|
| $\delta[n]$ | $1$ |
| $\delta[n - n_0]$ | $e^{-j\omega n_0}$ |
| $1 \quad (-\infty < n < \infty)$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$ |
| $a^n u[n] \quad (\lvert a\rvert < 1)$ | $\frac{1}{1-ae^{-j\omega}}$ |
| $u[n]$ | $\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$ |
| $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$ | $\frac{1}{(1-ae^{-j\omega})^2}$ |
| $\frac{\sin(\omega_c n)}{\pi n}$ | $X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$ |
| $x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$ | $\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$ |
| $e^{j\omega_0 n}$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$ |

# Common z-transform pairs

| Sequence | Transform | ROC |
|---|---|---|
| $\delta[n]$ | $1$ | All $z$ |
| $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| $\delta[n - m]$ | $z^{-m}$ | All $z$ except 0 or $\infty$ |
| $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| $-na^n u[-n-1]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| $\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$ | $\frac{1-a^N z^{-N}}{1-az^{-1}}$ | $\lvert z\rvert > 0$ |
| $\cos(\omega_0 n)u[n]$ | $\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| $r^n \cos(\omega_0 n)u[n]$ | $\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |